Inter (Part-II) 2016

Mathematics	Group-I	PAPER: II
Time: 30 Minutes	(OBJECTIVE TYPE)	Marks: 20

Note: Four possible answers, A, B, C and D to each question are given. The choice which you think is correct, fill that circle in front of that question with Marker or Pen ink in the answer-book. Cutting or filling two or more circles will result in zero mark in that question.

1-1- $\frac{d}{dx}e^{f(x)}$ **equals:**

(a) $e^{f'(x)}$ (b) $e^{f(x)}.f'(x)$ ✓

(c) $\frac{f'(x)}{e^x}$ (d) $\frac{e^{f(x)}}{f'(x)}$

2- **Distance between (1, 2) and (2, 1) is:**

(a) 1 (b) 2

(c) $\sqrt{2}$ ✓ (d) 0

3- $\frac{d}{dx}\tan h^{-1}x$**, equals:**

(a) $\frac{1}{1+x^2}$ (b) $\frac{1}{x^2-1}$

(c) $\frac{1}{1-x^2}$ ✓ (d) $\frac{1}{\sqrt{x^2-1}}$

4- $\int_0^{\pi/4}\frac{\sec^2 x}{1+\tan x}dx$**:**

(a) 1 (b) 2

(c) $\ln 2$ ✓ (d) $\ln\sqrt{2}$

5- **The expression** $\ln(x+\sqrt{x^2+1})$ **equals:**

(a) $\sin h^{-1}x$ ✓ (b) $\cos h^{-1}x$

(c) $\tan h^{-1}x$ (d) $\text{cosec}\, h^{-1}x$

6- $\int (2x+3)^{1/2}\, dx$, equals:

(a) $\frac{1}{2}(2x+3)^{1/2}+c$ (b) $\frac{2}{3}(2x+3)^{3/2}+c$

(c) $\frac{1}{3}(2x+3)^{1/2}+c$ (d) $\frac{1}{3}(2x+3)^{3/2}+c$ ✓

7- The line $y = mx + c$, will be tangent to the parabola $y^2 = 4ax$ if:

(a) $c = -am^2$ (b) $c = \frac{a}{m}$ ✓

(c) $c = a(1+m^2)$ (d) $c = \frac{m}{a}$

8- $(\hat{i} \times \hat{k}) \times \hat{j}$ equals:

(a) -1 (b) 0 ✓

(c) 1 (d) ∞

9- Solution set of inequality $2x < 3$ is:

(a) $(-\infty, \frac{3}{2})$ ✓ (b) $(\frac{3}{2}, \infty)$

(c) $(-\infty, \infty)$ (d) $\left(\frac{-3}{2}, \frac{3}{2}\right)$

10- If α, β, γ be the direction angles of a vector then $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$ equals:

(a) -1 (b) 0

(c) 1 ✓ (d) 2

11- The differential co-efficient of $e^{\sin x}$ is:

(a) $e^{\sin x} . \cos x$ ✓ (b) $e^{\sin x} . \sin x$

(c) $e^{\cos x} . \cos x$ (d) $\sin x . e^{\sin x - 1}$

12- Equation of the line parallel to $x + 3y - 9 = 0$ is:

(a) $3x - y - 9 = 0$ (b) $3x + 9y + 7 = 0$ ✓

(c) $2x - 6y - 18 = 0$ (d) $x - 3y + 9 = 0$

13- Length of latus rectum of ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is:

(a) $\frac{9}{2}$ ✓ (b) $\frac{9}{4}$

(c) $\frac{16}{9}$ (d) $\frac{9}{16}$

14- The solution of differential equation $\frac{dy}{dx} = \sec^2 x$, is :

(a) $y = \cos x + c$ (b) $y = \sec x + c$
(c) $y = \cos^2 x + c$ (d) $y = \tan x + c$ ✓

15- If $f(x) = 2^x$, then $f'(x)$ equals:

(a) 2^{x-1} (b) $2^x \ln 2$ ✓
(c) $\frac{2^x}{\ln 2}$ (d) $\frac{\ln 2}{2^x}$

16- If $f'(c) = 0$, then $f(x)$ has relative maxima at $x = c$, if:

(a) $f''(c) < 0$ ✓ (b) $f''(c) = 0$
(c) $f''(c) > 0$ (d) $f''(c) \geq 1$

17- If $y = \ln(\sin x)$, then $\frac{dy}{dx}$ equals:

(a) $\tan x$ (b) $\cot x$ ✓
(c) $-\tan x$ (d) $-\cot x$

18- Antiderivative of $\cot x$, equals:

(a) $\ln(\cos x) + c$ (b) $\ln(\sin x) + c$ ✓
(c) $-\text{cosec}^2 x + c$ (d) $\ln(\sec x) + c$

19- $ax + by + c = 0$, will represent equation of straight line parallel to y-axis if:

(a) $a = 0$ (b) $b = 0$ ✓
(c) $c = 0$ (d) $a = 0, c = 0$

20- $\lim_{x \to 2} \frac{x^5 - 32}{x - 2}$ equals:

(a) 40 (b) 60
(c) 80 ✓ (d) 120

Inter (Part-II) 2016

Mathematics	Group-I	PAPER: II
Time: 2.30 Hours	(SUBJECTIVE TYPE)	Marks: 80

SECTION-I

2. Write short answers to any EIGHT (8) questions: 16

(i) Find fof(x) if $f(x) = \sqrt{x+1}$.

Ans

$$fof(x) = f[f(x)] = f(\sqrt{x+1})$$
$$= \sqrt{(\sqrt{x+1}) + 1}$$

(ii) Evaluate $\underset{x\to\pi}{\text{limit}} \dfrac{\sin x}{\pi - x}$.

Ans

$$\lim_{x\to\pi} \frac{\sin x}{\pi - x}$$
$$= \lim_{x\to\pi} \frac{\sin(\pi - x)}{\pi - x} = 1$$

(iii) Discuss continuity of f(x) at 3 when $f(x) = \begin{cases} x-1, & \text{if } x < 3 \\ 2x+1, & \text{if } 3 \le x \end{cases}$

Ans As, $f(x) = 2x + 1$

$f(3) = 2(3) + 1 = 7$

1^{st} condition is satisfied

And;

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} (x-1) = 3 - 1 = 2$$
$$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} (2x+1) = 6 + 1 = 7$$

$\because \lim_{x\to3^-} f(x) \neq \lim_{x\to3^+} f(x)$

i.e., 2^{nd} condition is not satisfied

$\therefore \lim_{x\to3} f(x)$ does not exist.

Hence f(x) is not continuous at x = 3.

(iv) Differentiate $f(x) = x^2$ by definition.

Ans

$$f(x+\delta x) = (x+\delta x)^2$$
$$f(x+\delta x) - f(x) = (x+\delta x)^2 - x^2$$
$$= x^2 + 2x\,\delta x + (\delta x)^2 - x^2$$
$$= 2x\,\delta x + (\delta x)^2$$

$$= (2x + \delta x)\,\delta x$$

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{(2x+\delta x)\delta x}{\delta x} = 2x + \delta x$$

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} (2x + \delta x) = 2x$$

i.e., $f'(x) = 2x$

(v) Differentiate $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$ w.r.t x.

Ans $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 = (\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2(\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)$

$$= x + \frac{1}{x} - 2$$

$$= x + x^{-1} - 2$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(x + x^{-1} - 2)$$

$$= \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1}) - \frac{d}{dx}(2)$$

$$= 1 - x^{-2} - 0$$

$$\frac{dy}{dx} = 1 - \frac{1}{x^2}$$

(vi) Find $\frac{dy}{dx}$ if $3x + 4y + 7 = 0$.

Ans $3x + 4y + 7 = 0$

Differentiating the equation w.r.t 'x'.

$$\frac{d}{dx}(3x + 4y + 7) = 0$$

$$\frac{d}{dx}(3x) + \frac{d}{dx}(4y) + \frac{d}{dx}(7) = 0$$

$$3(1) + 4\frac{dy}{dx} + 0 = 0$$

$$4\frac{dy}{dx} = -3$$

$$\frac{dy}{dx} = \frac{-3}{4}$$

(vii) Differentiate sin x w.r.t. cot x.

Ans Let $u = \sin x$

and $v = \cot x$

Thus, $\frac{du}{dx} = \frac{d}{dx}(\sin x)$

$= \cos x$

$\frac{dv}{dx} = \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

By Chain Rule:

$$\frac{du}{dv} = \frac{du}{dx} \cdot \frac{dx}{dt}$$

$$= \cos x \left(\frac{1}{-\operatorname{cosec}^2 x}\right)$$

$$\frac{du}{dv} = -\cos x \sin^2 x$$

(viii) Differentiate $\cos\sqrt{x} + \sqrt{\sin x}$ w.r.t. x.

Ans Let $y = \cos\sqrt{x} + \sqrt{\sin x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos x^{1/2} + (\sin x)^{1/2}\right)$$

$$= \frac{d}{dx}(\cos x^{1/2}) + \frac{d}{dx}(\sin x)^{1/2}$$

$$= -\sin x^{1/2} \cdot \frac{d}{dx}(x)^{1/2} + \frac{1}{2}(\sin x)^{-1/2} \cdot \cos x$$

$$= -\sin\sqrt{x}\left(\frac{1}{2}\right)x^{-1/2} + \frac{1}{2}\frac{1}{(\sin x)^{1/2}}\cos x$$

$$= \frac{-\sin\sqrt{x}}{2\sqrt{x}} + \frac{\cos x}{2\sqrt{\sin x}}$$

$$= \frac{1}{2}\left(\frac{-\sin\sqrt{x}}{\sqrt{x}} + \frac{\cos x}{\sqrt{\sin x}}\right)$$

(ix) Find f′(x) if $f(x) = \ln(e^x + e^{-x})$.

Ans

$$f(x) = \ln(e^x + e^{-x})$$

$$f'(x) = \frac{d}{dx}[\ln(e^x + e^{-x})]$$

$$= \frac{1}{e^x + e^{-x}}\frac{d}{dx}(e^x + e^{-x})$$

$$= \frac{1}{e^x + \frac{1}{e^x}}(e^x + e^{-x})(-1)$$

$$= \frac{1}{\frac{e^x . e^x + 1}{e^x}} \cdot \left[\frac{e^x + e^{-x}(-1)}{e^x}\right] e^x - e^{-x}$$

$$= e^x - \frac{1}{e^x}$$

$$= \frac{e^x}{e^{2x}+1}\left[\frac{e^{2x}-1}{e^x}\right]$$

$$= \frac{e^{2x}-1}{e^{2x}+1}$$

(x) Find $\frac{dy}{dx}$ if $y = e^{-x}(x^3 + 2x^2 + 1)$.

Ans $\frac{dy}{dx} = \frac{d}{dx}[e^{-x}(x^3 + 2x^2 + 1)]$

$$= (e^{-x})\frac{d}{dx}(x^3 + 2x^2 + 1) + (x^3 + 2x^2 + 1)\frac{d}{dx}(e^{-x})$$
$$= e^{-x}(3x^2 + 4x + 0) + (x^3 + 2x^2 + 1)(e^{-x})(-1)$$
$$= e^{-x}(3x^2 + 4x) - (x^3 + 2x^2 + 1)(e^{-x})$$
$$= e^{-x}[3x^2 + 4x - x^3 - 2x^2 - 1]$$
$$= e^{-x}(-x^3 + x^2 + 4x - 1)$$

(xi) Find y_4 if $y = \sin 3x$.

Ans

$$\frac{dy}{dx} = \frac{d}{dx}(\sin 3x)$$

$$\frac{dy}{dx} = \cos 3x \,.\, 3$$
$$= 3\cos 3x$$

$$\frac{d^2y}{dx^2} = 3\frac{d}{dx}(\cos 3x)$$
$$= (3)(-\sin 3x)(3)$$
$$= -9\sin 3x$$

$$\frac{d^3y}{dx^3} = -9\frac{d}{dx}(\sin 3x)$$
$$= -9(\cos 3x)(3)$$
$$= -27\cos 3x$$

$$\frac{d^4y}{dx^4} = -27\frac{d}{dx}(\cos 3x)$$
$$= -27(-\sin 3x)(3)$$

$$\frac{d^4y}{dx^4} = 81\sin 3x$$

As $\frac{d^4y}{dx^4} = y_4$

So, $\boxed{y_4 = 81\sin 3x}$

(xii) Expand cos x by Maclaurin's series expansion.

Ans

$f(x) = \cos x$

$f(0) = \cos(0) = 1$ (i)

$f'(x) = -\sin x$

$f'(0) = -\sin(0) = 0$ (ii)

$f''(x) = -\cos x$

$f''(0) = -\cos(0) = -1$ (iii)

$f'''(x) = -(-\sin x)$

$= \sin x$

$= \sin(0) = 0$ (iv)

$f^4(x) = \cos x$

$= \cos(0) = 1$ (v)

$f^5(x) = -\sin x$

$= -\sin(0) = 0$ (vi)

$f^6(x) = -\cos x$

$= -\cos(0) = -1$ (vii)

The Maclaurin series expansion of f(x) is:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \frac{f^5(0)}{5!}x^5 + \frac{f^6(0)}{6!}x^6 + \ldots\ldots$$

Putting (i), (ii), (iii), (iv), (v), (vi) and (vii) above:

$$f(x) = 1 + (0)x + \frac{(-1)}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 + \frac{(-1)}{6!}x^6 + \ldots$$

$$\therefore \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

3. Write short answers to any EIGHT (8) questions: 16

(i) Find δy, if $y = x^2 + 2x$ when x changes from 2 to 1.8.

Ans

$y = x^2 + 2x$

When x changes from 2 to 1.8

$\therefore \quad x = 2$

$\Delta x = 1.8 - 2 = -0.2$

$y = x^2 + 2x$

$= (2)^2 + 2(2) = 4 + 4$

$= 8$

$y = x^2 + 2x$

For change Δx in x

$y + \Delta y = (x + \Delta x)^2 + 2(x + \Delta x)$

$\Delta y = (x + \Delta x)^2 + 2(x + \Delta x) - y$

$\Delta y = (2 - 0.2)^2 + 2(2 - 0.2) - 8$

$\Delta y = 3.24 + 3.6 - 8$

$= -1.16$

(ii) Use differential, find the value of $(31)^{1/5}$.

Ans Let

$y = x^{1/5}$; and $x = 32$

$y = (32)^{1/5} = (2^5)^{1/5} = 2$; $\therefore dx = 31 - 32 = -1$

$$\frac{dy}{dx} = \frac{d}{dx}(x)^{1/5}$$

$$= \frac{1}{5}x^{-4/5}$$

$$\therefore \quad dy = \frac{1}{5x^{4/5}}dx$$

$$= \frac{1}{5[(32)^{1/5}]^4}(-1) \Rightarrow \frac{1}{5[(2^5)]^{4/5}}(-1)$$

$$= \frac{-1}{5(16)}$$

$$= \frac{-1}{80}$$

$$= -0.013$$

$$(31)^{1/5} = y + dy$$

$$= 2 + (-0.013)$$

$$= 1.987$$

(iii) Evaluate $\int x(\sqrt{x} + 1)\,dx$.

Ans $\int x(\sqrt{x}+1)\,dx = \int x \,.\, x^{1/2}\,dx + \int x\,dx$

$$= -\int x^{3/2}\,dx + \frac{x^2}{2} + c$$

$$= \frac{x^{5/2}}{\frac{5}{2}} + \frac{x^2}{2} + c$$

$$= \frac{2}{5}x^{5/2} + \frac{x^2}{2} + c$$

(iv) Evaluate $\int \tan^2 x\,dx$.

Ans $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx$

$$= \int \sec^2 x\,dx - \int 1\,dx$$

$$= \int \sec^2 x\,dx - \int dx$$

$$= \tan x - x + c$$

(v) **Evaluate $\int \frac{1}{x \ln x} dx$.**

Ans Let $\ln x = t$

$\therefore \frac{1}{x} dx = dt$

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{\ln x} \cdot \frac{1}{x} dx$$
$$= \int \frac{1}{t} dt$$
$$= \ln |t| + c$$
$$= \ln |\ln x| + c$$

(vi) **Evaluate $\int e^{2x} (-\sin x + 2\cos x)\, dx$.**

Ans $\int e^{2x} (-\sin x + 2\cos x)\, dx = -\int e^{2x} \sin x + 2 \int e^{2x} \cos x\, dx$

Integrating first integral by parts:

$$= -e^{2x} \int \sin x\, dx + \int \left[\frac{d}{dx}(e^{2x}) \cdot \int \sin x\, dx\right] dx + 2 \int e^{2x} \cos x\, dx + c$$
$$= -e^{2x} (-\cos x) + \int 2e^{2x} \cos x\, dx + 2 \int e^{2x} \cos x\, dx + c$$
$$= e^{2x} \cos x + c$$

(vii) **Evaluate $\int_{-2}^{0} \frac{1}{(2x-1)^2} dx$.**

Ans
$$\int_{-2}^{0} \frac{1}{(2x-1)^2} dx = \frac{2}{2} \int_{-2}^{0} \frac{1}{(2x-1)^2} dx$$
$$= \frac{1}{2} \int_{-2}^{0} (2x-1)^{-2} \cdot (2)\, dx$$
$$= \frac{1}{2} \left[\frac{(2x-1)^{-1}}{-1}\right]_{-2}^{0}$$
$$= \frac{-1}{2} [(2x-1)^{-1}]_{-2}^{0}$$
$$= \frac{-1}{2} \left[\frac{1}{2x-1}\right]_{-2}^{0}$$
$$= \frac{-1}{2} \left[\frac{1}{2(0)-1} - \frac{1}{2(-2)-1}\right]$$
$$= \frac{-1}{2} \left[-1 + \frac{1}{5}\right]$$

$$= \frac{-1}{2}\left[\frac{-4}{5}\right]$$

$$= \frac{2}{5}$$

(viii) Evaluate $\int_1^2 \frac{x}{x^2+2}\,dx$.

Ans

$$\int_1^2 \frac{x}{x^2+2}\,dx = \frac{2}{2}\int_1^2 \frac{x}{x^2+2}$$

$$= \frac{1}{2}\int_1^2 \frac{2x}{x^2+2}$$

$$= \frac{1}{2}\left[\ln|x^2+2|\right]_1^2$$

$$= \frac{1}{2}\left[\ln(2^2+2) - \ln(1^2+2)\right)$$

$$= \frac{1}{2}\left[\ln|6| - \ln|3|\right]$$

$$= \frac{1}{2}\left[\ln(2\times 3) - \ln(3)\right]$$

$$= \frac{1}{2}\left[\ln(2) + \ln(3) - \ln(3)\right]$$

$$= \frac{\ln(2)}{2}$$

(ix) Find the area above the x-axis and under the curve, $y = 5 - x^2$ from $x = -1$ to $x = 2$.

Ans

$$\text{Area} = \int_{-1}^{2} (5 - x^2)\,dx$$

$$= 5\int_{-1}^{2} dx - \int_{-1}^{2} x^2\,dx$$

$$= 5\,[x]_{-1}^{2} - \left[\frac{x^3}{3}\right]_{-1}^{2}$$

$$= 5[2-(-1)] - \frac{1}{3}[8-(-1)^3]$$

$$= 5(3) - \frac{1}{3}(9)$$

$$= 15 - 3$$

$$= 12 \text{ sq. units}$$

(x) Solve the differential equation $y\,dx + x\,dy = 0$.

Ans

$$y\,dx + x\,dy = 0$$
$$y\,dx = -x\,dy$$
$$\frac{dx}{x} = \frac{-dy}{y}$$
$$\frac{dx}{x} + \frac{dy}{y} = 0$$
$$\int\frac{dx}{x} + \int\frac{dy}{y} = \ln|c|$$
$$\ln|x| + \ln|y| = \ln|c|$$
$$\ln|x\,y| = \ln|c|$$
$$xy = c$$

(xi) What is convex region?

Ans If the line segment obtained by joining any two points of a region lies entirely within the region, then the region is called convex.

(xii) Graph the solution set of linear inequality $3x + 7y \geq 21$ by shading.

Ans

$$3x + 7y \geq 21 \quad \text{(i)}$$
$$3x + 7y = 21 \quad \text{(ii)}$$

Putting $x = 0$, in (ii)

$$0 + 7y = 21 \quad \Rightarrow \quad y = 3$$

$\therefore$ (0, 3) is a point on (ii)

Putting $y = 0$ in (ii)

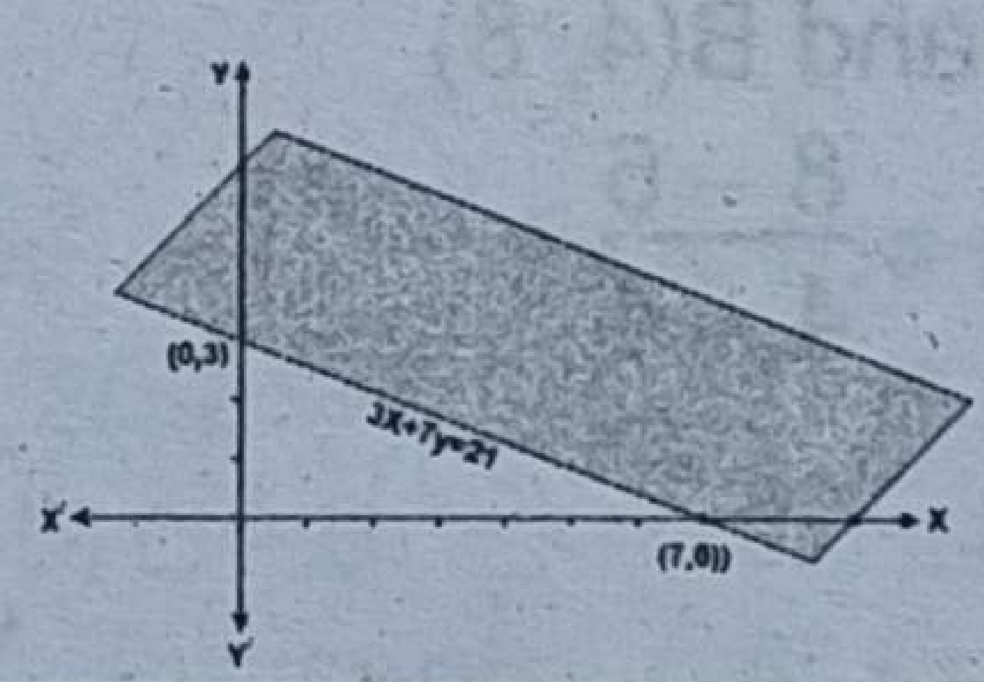
$$3x + 0 = 21 \quad \Rightarrow \quad x = 7$$

$\therefore$ (7, 0) is another point on (ii)

Putting $x = 0, y = 0$ in (i)

$$0 + 0 > 21$$
$$\therefore \quad 0 > 21$$

which is false. Hence solution region of (i) does not lie on the origin-side of (i).



4. Write short answers to any NINE (9) questions: 18

(i) Find h such that A(−1, h), B(3, 2) and C(7, 3) are collinear.

Ans A, B and C are collinear, if:

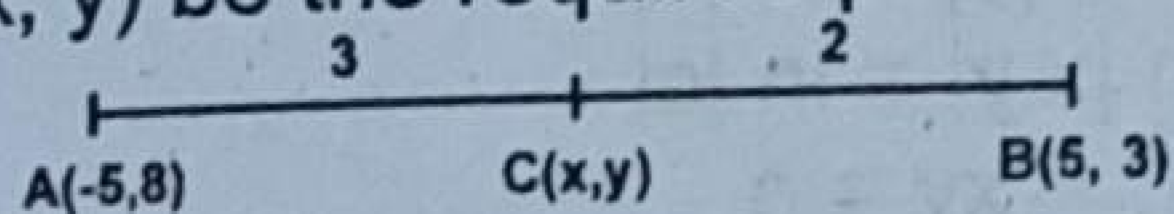
$$\begin{vmatrix} -1 & h & 1 \\ 3 & 2 & 1 \\ 7 & 3 & 1 \end{vmatrix} = 0$$

Expanding by row 1:

$$-1(2-3) - h(3-7) + 1(9-14) = 0$$
$$1 + 4h - 5 = 0$$
$$4h = 4$$
$$h = 1$$

(ii) Find the point three-fifth of the way along line segment from A(–5, 8) to B(5, 3).

Ans Let C(x, y) be the required point.



Given condition

$$AC = \frac{3}{5}AB$$
$$= \frac{3}{5}(AC + CB) \Rightarrow \frac{3}{5}AC + \frac{3}{5}CB$$
$$AC - \frac{3}{5}AC = \frac{3}{5}CB$$
$$\left(1 - \frac{3}{5}\right)AC = \frac{3}{5}CB \Rightarrow \frac{2}{5}AC = \frac{3}{5}CB$$
$$\frac{AC}{CB} = \frac{3}{5} \times \frac{5}{2} = \frac{3}{2} \qquad \Rightarrow AC : CB = 3 : 2$$

Coordinates of C are:

$$x = \frac{3(5) + (2)(-5)}{3+2} \Rightarrow \frac{15-10}{5} = \frac{5}{5} = 1$$
$$y = \frac{(3)(3) + (2)(8)}{3+2} = \frac{9+16}{5} = \frac{25}{5} = 5$$

$\therefore$ C(1, 5)

(iii) Find slope and inclination of line joining points (4, 6), (4, 8).

Ans Let A(4, 6) and B(4, 8)

$$\text{Slope of } AB = m = \frac{8-6}{4-4}$$
$$= \frac{2}{0}$$
$$= \infty \text{ undefined}$$
$$m = \tan\theta = \infty \Rightarrow \text{Inclination}$$
$$= \theta$$

$= \tan^{-1}(\infty)$
$= 90^\circ$

(iv) Find measure of the angle between the lines represented by $x^2 - xy - 6y^2 = 0$.

Ans Here $a = 1, h = \frac{-1}{2}, b = -6$

If θ is measure of the angle between the given lines, then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$
$$= \frac{2\sqrt{\frac{1}{4} + 6}}{-5}$$
$$= -1$$
$$= \theta$$
$$= 135^\circ$$

Acute angle between the lines $= 180^\circ - \theta$
$= 180^\circ - 135^\circ$
$= 45^\circ$

(v) Find the distance from the point P(6, –1) to the line $6x - 4y + 9 = 0$.

Ans
$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$
$$= \frac{|6(6) + (-4)(-1) + 9|}{\sqrt{(6)^2 + (-4)^2}}$$
$$= \frac{49}{\sqrt{52}}$$
$$= \frac{49}{2\sqrt{3}}$$

(vi) Find the centre and radius of the circle $x^2 + y^2 + 12x - 10y = 0$.

Ans
$$x^2 + y^2 + 12x - 10y = 0$$
$$x^2 + y^2 + 2(6)x + 2(-5)y + 0 = 0$$

$\therefore$ $g = 6, f = -5, c = 0$

$\therefore$ centre $(-g, -f)$ = centre $(-6, 5)$

and Radius $= \sqrt{g^2 + f^2 - c}$
$= \sqrt{(6)^2 + (-5)^2 - 0}$
$= \sqrt{36 + 25} = \sqrt{61}$

(vii) Find the coordinate of the points of intersection of the line $x + 2y = 6$ with the circle $x^2 + y^2 - 2x - 2y - 39 = 0$.

Ans

$$x + 2y = 6 \quad \text{(i)}$$

$$x^2 + y^2 - 2x - 2y - 39 = 0 \quad \text{(ii)}$$

From (i),

$$x = 6 - 2y \quad \text{(iii)}$$

By putting (iii) in (ii),

$$(6 - 2y)^2 + y^2 - 2(6 - 2y) - 2y - 39 = 0$$

$$36 - 24y + 4y^2 + y^2 - 12 + 4y - 2y - 39 = 0$$

$$5y^2 - 22y - 15 = 0$$

$$y = \frac{-(-22) \pm \sqrt{(-22)^2 - 4(5)(-15)}}{2(5)}$$

$$= \frac{22 \pm \sqrt{484 + 300}}{10}$$

$$= \frac{22 \pm \sqrt{784}}{10}$$

$$= \frac{22 \pm 28}{10}$$

$$y = \frac{20 + 28}{10} = \frac{48}{10}; \qquad y = \frac{20 - 28}{10} = \frac{-8}{10}$$

$$y = \frac{24}{5} \quad ; \qquad y = \frac{-4}{5}$$

(viii) Find the focus and vertex of the parabola $y^2 = 8x$.

Ans

$$y^2 = 8x$$

Comparing it with $y^2 = 4ax$, as it is parabola.

$$4a = 8$$

$$a = \frac{8}{4}$$

So, $a = 2$

Coordinates of focus : $F(a, 0) = F(2, 0)$

Coordinate of vertex : $V(0, 0)$

(ix) Find an equation of the vertical line through $(-5, 3)$.

Ans

$$\text{Slope} = m = \tan\theta$$

$$= \tan 90^\circ$$

$$= \infty$$

Equation of the line through $(-5, 3)$ is:

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \infty(x + 5)$$

$$\frac{y-3}{x+5} = \infty(x+5) = 0$$

$$\Rightarrow x + 5 = 0$$

$$\therefore \quad x = -5$$

(x) If O is the origin and $\overrightarrow{OP} = \overrightarrow{AB}$, find the point P when A and B are (–3, 7) and (1, 0), respectively?

Ans Let $P(x, y), O(0, 0), A(-3, 7), B(1, 0)$

$$\overrightarrow{OP} = \overrightarrow{AB} \quad \text{(i)}$$

$$\overrightarrow{OP} = (x-0)\,\underline{i} + (y-0)\,\underline{j} = x\,\underline{i} + y\,\underline{j} \quad \text{(ii)}$$

$$\overrightarrow{AB} = (1+3)\,\underline{i} + (0-7)\,\underline{j} = 4i - 7\underline{j} \quad \text{(iii)}$$

By putting (ii) and (iii) in (i), we get

$$x\,\underline{i} + y\,\underline{j} = 4\underline{i} - 7\underline{j}$$

$$\Rightarrow \quad x = 4, \quad y = -7$$

$$\therefore \quad P(4, -7)$$

(xi) Find a vector whose magnitude is 4 and is parallel to $2\overline{i} - 3\overline{j} + 6\overline{k}$.

Ans Let $\underline{v} = 2\underline{i} - 3\underline{j} + 6\underline{k}$

$$\therefore \quad |\underline{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2}$$

$$= \sqrt{4 + 9 + 36}$$

$$= \sqrt{49}$$

$$= 7$$

$$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} = \frac{2\underline{i} - 3\underline{j} + 6\underline{k}}{7}$$

$$\hat{\underline{v}} = 4\left(\frac{2\underline{i} - 3\underline{j} + 6\underline{k}}{7}\right)$$

(xii) Find a vector perpendicular to each of the vectors $\underline{a} = 2\overline{i} - \overline{j} - \overline{k}$ and $\underline{b} = 4\overline{i} + 2\overline{j} - \overline{j} - \overline{k}$.

Ans A vector perpendicular to both the vectors $\underline{a}$ and $\underline{b}$ is $\underline{a} \times \underline{b}$.

$$\underline{b} = 4\overline{i} + \overline{j} - \overline{k}$$

$$\therefore \quad \underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & -1 \\ 4 & 1 & -1 \end{vmatrix}$$

$$= i(1+1) - j(-2+4) + k(2+4)$$

$$= 2i - 2j + 6k$$

Verification:

$$\underline{a}\cdot\underline{a}\times\underline{b} = (2\underline{i} - \underline{j} - \underline{k})(2\underline{i} - 2\underline{j} + 6\underline{k})$$
$$= 2(2) + (-1)(-2) + (-1)(6)$$
$$= 6 - 6 = 0$$

and $\underline{b}\cdot\underline{a}\times\underline{b} = (4\underline{i} + 1\underline{j} - \underline{k})\cdot(2i - 2\underline{j} + 6\underline{k})$

$$= 4(2) + (1)(-2) + (-1)(6)$$
$$= 8 - 8 = 0$$

Hence $\underline{a}\times\underline{b}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$.

(xiii) Find the value of $3\bar{j}\cdot\bar{k}\times\bar{i}$.

Ans

$$(0\underline{i} + 3\underline{j} + 0\underline{k})\cdot[(0\underline{i} + 0\underline{j} + \underline{k})\times(\underline{i} + 0\underline{j} + 0\underline{k})]$$

$$= \begin{vmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0 - 3(0 - 1) + 0 = 3$$

SECTION-II

NOTE: Attempt any Three (3) questions.

Q.5.(a) Evaluate $\lim_{\theta\to 0}\dfrac{1-\cos p\theta}{1-\cos q\theta}$. **(5)**

Ans

$$\lim_{\theta\to 0}\frac{1-\cos p\theta}{1-\cos q\theta} = \lim_{\theta\to 0}\frac{2\sin^2\frac{p\theta}{2}}{2\sin^2\frac{q\theta}{2}}$$

$$= \lim_{\theta\to 0}\left[2\sin^2\frac{p\theta}{2} \div 2\sin^2\frac{q\theta}{2}\right]$$

$$= \lim_{\theta\to 0}\left[\frac{\sin^2\frac{p\theta}{2}}{\frac{p^2\theta^2}{4}\cdot\frac{4}{p^2\theta^2}} \div \frac{\sin^2\frac{q\theta}{2}}{\frac{q^2\theta^2}{4}\cdot\frac{4}{q^2\theta^2}}\right]$$

$$= \lim_{\theta\to 0}\left[\left(\frac{\sin\frac{p\theta}{2}}{\frac{p\theta}{2}}\right)^2\frac{p^2\theta^2}{4} \div \left(\frac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}}\right)^2\cdot\frac{q^2\theta^2}{4}\right]$$

$$= \lim_{\theta\to 0}\left[\left(\frac{\sin\frac{p\theta}{2}}{\frac{p\theta}{2}}\right)^2\cdot\frac{p^2\theta^2}{4}\cdot\frac{1}{\left(\frac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}}\right)^2}\cdot\frac{4}{q^2\theta^2}\right]$$

$$= \frac{p^2}{q^2}\left[\lim_{\theta \to 0} \frac{\sin \frac{p\theta}{2}}{\frac{p\theta}{2}}\right]^2 \cdot \frac{1}{\left[\lim_{\theta \to 0} \frac{\sin \frac{q\theta}{2}}{\frac{q\theta}{2}}\right]^2}$$

$$= \frac{p^2}{q^2} \cdot (1)^2 \cdot \frac{1}{(1)^2}$$

$$= \frac{p^2}{q^2}$$

(b) Differentiate $\sin\sqrt{\frac{1+2x}{1+x}}$ w.r.t. x. (5)

Ans Let, $y = \sin\sqrt{\frac{1+2x}{1+x}}$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin\sqrt{\frac{1+2x}{1+x}}\right]$$

$$= \cos\sqrt{\frac{1+2x}{1+x}} \cdot \frac{d}{dx}\left[\frac{(1+2x)^{1/2}}{(1+x)^{1/2}}\right]$$

$$= \cos\sqrt{\frac{1+2x}{1+x}}\left[\frac{(1+x)^{1/2}\frac{d}{dx}(1+2x)^{1/2} - (1+2x)^{1/2}\frac{d}{dx}(1+x)^{1/2}}{[(1+x)^{1/2}]^2}\right]$$

$$= \cos\sqrt{\frac{1+2x}{1+x}}\left[\frac{(1+x)^{1/2} \cdot \frac{1}{2}(1+2x)^{1/2-1}\frac{d}{dx}(1+2x) - (1+2x)^{1/2}\frac{1}{2}(1+x)^{-1/2}(0+1)}{1+x}\right]$$

$$= \cos\sqrt{\frac{1+2x}{1+x}}\left[\frac{(1+x)^{1/2} \cdot \frac{1}{2}(1+2x)^{-1/2}(2) - (1+2x)^{1/2} \cdot (1+x)^{-1/2} \cdot 1}{1+x}\right]$$

$$= \cos\sqrt{\frac{1+2x}{1+x}}\left[\frac{\frac{(1+x)^{1/2}}{(1+2x)^{1/2}} \cdot \frac{(1+2x)^{1/2}}{2(1+x)^{1/2}}}{1+x}\right]$$

$$= \cos\sqrt{\frac{1+2x}{1+x}}\left[\frac{\frac{2(1+x)^{1/2} \cdot (1+x)^{1/2} - (1+2x)^{1/2} \cdot (1+2x)^{1/2}}{2(1+2x)^{1/2}(1+x)^{1/2}}}{1+x}\right]$$

$$= \cos\sqrt{\frac{1+2x}{1+x}}\left[\frac{2(1+x) - (1+2x)}{2\sqrt{1+2x}\,(1+x)^{1/2} \cdot (1+x)}\right]$$

$$= \cos\sqrt{\frac{1+2x}{1+x}}\left[\frac{2+2x-1-2x}{2\sqrt{1+2x}\,(1+x)^{3/12}}\right]$$

$$= \frac{\cos\sqrt{\frac{1+2x}{1+x}}}{2\sqrt{1+2x}\,(1+x)^{3/2}}$$

Q.6.(a) Evaluate $\int \frac{\cos x}{\sin x \ln(\sin x)} dx$. (5)

Ans Let $\ln \sin x = t$

By taking derivative both sides, we get

$$\frac{1}{\sin x}.\cos x\, dx = dt$$

$$\frac{\cos x}{\sin x} dx = dt$$

Now, integrate both sides

$$\int \frac{\cos x}{\sin x} dx = \int dt$$

$$\int \frac{\cos x}{\sin x \ln \sin x} dx = \int \frac{1}{\ln \sin x} \frac{\cos x}{\sin x} dx$$

$$= \int \frac{1}{t} dt$$

$$= \ln|t| + c$$

$$= \ln|\ln \sin x| + c$$

(b) Find distance between $3x - 4y + 3 = 0$ and $3x - 4y + 7 = 0$. Also find equation of parallel line lying midway between them. (5)

Ans

$$3x - 4y + 3 = 0 \quad \text{(i)}$$

$$3x - 4y + 7 = 0 \quad \text{(ii)}$$

Put $x = 0$ in (i)

$$3(0) - 4y + 3 = 0$$

$$-4y = -3$$

$$y = \frac{3}{4}$$

Put $x = 0$ in (ii)

$$3(0) - 4y + 7 = 0$$

$$0 - 4y = -7$$

$$y = \frac{7}{4}$$

Hence $(0, \frac{3}{4})$ is a point on (i) and $(0, \frac{7}{4})$ is a point on (ii).

Distance of $(0, \frac{3}{4})$ from (ii) is:

$$d = \frac{|a x_1 + b y_1 + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{(3)(0) + (-4)\left(\frac{3}{4}\right) + 7}{\sqrt{(3)^2 + (-4)^2}} = \frac{4}{5}$$

Midpoint of $(0, \frac{3}{4})$ and $(0, \frac{7}{4})$ is:

$$\left(\frac{0+0}{2}, \frac{\frac{3}{4}+\frac{7}{4}}{2}\right) = \left(0, \frac{5}{4}\right)$$

From (i): $-4y = -3x - 3$

$\Rightarrow \quad y = \frac{3}{4}x + \frac{3}{4}$

$\Rightarrow \quad m = \frac{3}{4}$

Equation of line through $(0, \frac{5}{4})$ with slope $= \frac{3}{4}$ is:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{5}{4} = \frac{3}{4}(x - 0)$$

$$\frac{3}{4}x - y + \frac{5}{4} = 0$$

$\Rightarrow \quad 3x - 4y + 5 = 0$

Q.7.(a) Evaluate $\int_2^3 \frac{3x^2 - 2x + 1}{(x-1)(x^2+1)} dx$. **(5)**

Ans Partial Fractions are:

$$\frac{3x^2 - 2x + 1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+1} \quad \text{(i)}$$

$$3x^2 - 2x + 1 = A(x^2 + 1) + (Bx + C)(x - 1) \quad \text{(ii)}$$

For A, let $x - 1 = 0 \Rightarrow x = 1$

Putting it in (ii):

$$3 - 2 + 1 = A(1 + 1) + 0$$

$$2 = 2A$$

$\Rightarrow \quad A = 1$

Expanding (ii)

$3x^2 - 2x + 1 = Ax^2 + A + Bx^2 - Bx + Cx - C$

$3x^2 - 2x + 1 = (A + B)x^2 + (-B + C)x + (A - C)$

Comparing coefficient on both sides:

$A + B = 3$

$1 + B = 3$

$\boxed{B = 2}$

$-B + C = -2$

$-2 + C = -2$

$\boxed{C = 0}$

$A - C = 1$

$A - 0 = 1$

$\boxed{A = 1}$

By putting the values in (i), we get

$$\frac{3x^2 - 2x + 1}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{2x+0}{x^2+1}$$

$$= \frac{1}{x-1} + \frac{2x}{x^2+1}$$

$$\int_2^3 \frac{3x^2 - 2x + 1}{(x-1)(x^2+1)} = \int_2^3 \frac{dx}{x-1} + \int_2^3 \frac{2x}{x^2+1}\,dx$$

$$= [\ln|x-1|]_2^3 + [\ln|x^2+1|]_2^3$$

$$= \ln|3-1| - \ln|2-1| + \ln|9+1| - \ln|4+1|$$

$$= \ln(2) - \ln(1) + \ln(10) - \ln(5)$$

$$= \ln(2) - 0 + \ln(2 \times 5) - \ln(5)$$

$$= \ln(2) + \ln(2) + \ln(5) - \ln(5)$$

$$= 2\ln(2)$$

$$= \ln(2)^2$$

$$= \ln(4)$$

(b) Maximize $f(x, y) = x + 3y$ subject to the constraints: (5)

$2x + 5y \leq 30, \quad 5x + 4y \leq 20, \quad x \geq 0, \quad y \geq 0$

Ans $2x + 5y \leq 30$ (i)

$5x + 4y \leq 20$ (ii)

$2x + 5y = 30$ (iii)

$5x + 4y = 20$ (iv)

Put $x = 0$ in (iii)

$2(0) + 5y = 30$

$$5y = 30$$
$$y = 6$$

(0, 6) is a point on equation (iii),

Put x = 0 in (iv)

$$5(0) + 4y = 20$$
$$4y = 20$$
$$y = 5$$

(0, 5) a point on equation (iv),

Put y = 0 in (iii)

$$2x + 5(0) = 30$$
$$2x = 30$$
$$x = 15$$

(15, 0) is a point on equation (iii),

Put y = 0 in (iv)

$$5x + 4(0) = 20$$
$$5x = 20$$
$$x = 4$$

(4, 0) is a point on equation (iv),

By putting x = 0, y = 0 in (i).

$$0 + 0 < 30$$
$$0 < 30 \quad \text{True}$$

By putting x = 0, y = 0 in (ii).

$$0 + 0 < 20$$
$$0 < 20 \quad \text{True}$$

Q.8.(a) Find the length of the chord cut off from the line $2x + 3y = 13$ by the circle $x^2 + y^2 = 26$. (5)

Ans

$$2x + 3y = 13 \quad \text{(i)}$$
$$x^2 + y^2 = 26 \quad \text{(ii)}$$

From (i):

$$2x = 13 - 3y$$
$$\Rightarrow \quad x = \frac{13 - 3y}{2} \quad \text{(iii)}$$

By putting (iii) in (ii), we get

$$\left(\frac{13 - 3y}{2}\right)^2 + y^2 = 26$$
$$\frac{169 - 78y + 9y^2}{4} + y^2 = 26$$
$$\frac{169 - 78y + 9y^2 + 4y^2}{4} = 26$$

$$169 - 78y + 13y^2 = 104$$
$$13y^2 - 78y + 169 - 104 = 0$$
$$13y^2 - 78y + 65 = 0$$
$$13(y^2 - 6y + 5) = 0$$
$$y^2 - 6y + 5 = 0$$
$$y = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{36 - 20}}{2}$$
$$= \frac{6 \pm \sqrt{16}}{2}$$
$$y = \frac{6 \pm 4}{2}$$
$$y = \frac{6+4}{2} \quad ; \quad y = \frac{6-4}{2}$$
$$= \frac{10}{2} = 5 \quad ; \quad = \frac{2}{2} = 1$$

By putting these values in (iii)

$$x = \frac{13 - 3(5)}{2} = -1$$
$$x = \frac{13 - 3(1)}{2} = 5$$

So, (–1, 5) and (5, 1) are the points of intersection of (i) and (ii).
Thus, finally

$$\text{Length of chord} = l = \sqrt{(5+1)^2 + (1-5)^2}$$
$$= \sqrt{36 + 16}$$
$$= \sqrt{52}$$
$$= 2\sqrt{13}$$

(b) Prove that in any triangle ABC by vector method $a^2 = b^2 + c^2 - 2bc \cos A$. **(5)**

Ans Let the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ be along the sides BC, CA and AB of the triangle ABC as shown in the figure.

$$\therefore \quad \underline{a} + \underline{b} + \underline{c} = \underline{0}$$
$$\Rightarrow \quad \underline{a} = -(\underline{b} + \underline{c})$$

Now $\underline{a} \cdot \underline{a} = (\underline{b} + \underline{c}) \cdot (\underline{b} + \underline{c})$

$$\underline{a} \cdot \underline{a} = \underline{b} \cdot \underline{b} + \underline{b} \cdot \underline{c} + \underline{c} \cdot \underline{b} + \underline{c} \cdot \underline{c}$$
$$a^2 = b^2 + bc + bc + c^2$$

$$= b^2 + c^2 + 2bc\cos(\pi - A)$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

Q.9.(a) Show that the equation $9x^2 - 18x + 4y^2 + 8y - 23 = 0$ represents an ellipse. Find its elements (foci, vertices, directrices). (5)

Ans $9x^2 - 18x + 4y^2 + 8y - 23 = 0$ (1)

We complete the squares in (1) and it becomes

$$(9x^2 - 18x + 9) + (4y^2 + 8y + 4) - 36 = 0$$
$$9(x-1)^2 + 4(y+1)^2 = 36$$
$$\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1 \quad (2)$$

If we set $x - 1 = X$, $y + 1 = Y$ into (2), it becomes

$$\frac{X^2}{2^2} + \frac{Y^2}{3^2} = 1 \quad (3)$$

which is an ellipse with major axis along $X = 0$, *i.e.*, along the line $x - 1 = 0$ (*i.e.*, a line parallel to the y-axis).

Semi-major axis = 3,
Semi-minor axis = 2

$$c = \sqrt{9-4} = \sqrt{5}$$

Ecentricity $= \frac{c}{a} = \frac{\sqrt{5}}{3}$

Centre of (2) is $X = 0$, $Y = 0$
Or $x = 1$, $y = -1$ *i.e.*, $(1, -1)$ is centre of (1).
The foci of (2) are

$$X = 0, Y = \pm\sqrt{5}$$

i.e., $x - 1 = 0$, $y + 1 = \pm\sqrt{5}$
i.e., $(1, -1+\sqrt{5})$ and $(1, -1-\sqrt{5})$ are foci of (1).
Vertices of (2) are

$X = 0$, $Y = \pm 3$, *i.e.*, $x = 1$, $y = -1 \pm 3$
or $(1, -4)$ and $(1, 2)$ are the vertices of (1).

Directrices $= x = \pm\frac{c}{e^2}$

$$= \frac{\pm\sqrt{5}}{\left(\frac{\sqrt{5}}{3}\right)^2} = \frac{\pm\sqrt{5}}{\frac{5}{9}} = \pm\frac{9}{\sqrt{5}}$$

(b) Find volume of the tetrahedron whose vertices are: (5)

A(2, 1, 8), B(3, 2, 9), C(2, 1, 4), D(3, 3, 10)

Ans

$$\vec{AB} = (3-2)\,\underline{i} + (2-1)\,\underline{j} + (9-8)\,\underline{k} = \underline{i} + \underline{j} + \underline{k}$$

$$\vec{AC} = (2-2)\,\underline{i} + (1-1)\,\underline{j} + (4-8)\,\underline{k} = 0\underline{i} + 0\underline{j} - 4\underline{k}$$

$$\vec{AD} = (3-2)\,\underline{i} + (3-1)\,\underline{j} + (10-8)\,\underline{k} = \underline{i} + 2\underline{j} + 2\underline{k}$$

$\therefore$ Volume of the tetrahedron $= \frac{1}{6}[\vec{AB}\ \vec{AC}\ \vec{AD}]$

$$= \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= \frac{1}{6}[1(0+8) - 1(0+4) + 1(0, 0)]$$

$$= \frac{1}{6}[8-4]$$

$$= \frac{1}{6}[4(2-1)]$$

$$= \frac{4}{6} = \frac{2}{3}$$