Inter (Part-II) 2016

Mathematics	Group-II	PAPER: II
Time: 30 Minutes	(OBJECTIVE TYPE)	Marks: 20

Note: Four possible answers, A, B, C and D to each question are given. The choice which you think is correct, fill that circle in front of that question with Marker or Pen ink in the answer-book. Cutting or filling two or more circles will result in zero mark in that question.

1-1- **$\int_{-\pi}^{\pi} \sin x \, dx$ is equal to:**

(a) 0 √ (b) 6
(c) 8 (d) 16

2- **If a straight line is parallel to y-axis, then its slope is:**

(a) –1 (b) 0
(c) 1 (d) Undefined √

3- **Centre of the circle $(x - 1)^2 + (y + 3)^2 = 3$ is:**

(a) (–1, –3) (b) (–1, 3)
(c) (1, 3) (d) (1, –3) √

4- **A linear equation in two variables represents:**

(a) Circle (b) Ellipse
(c) Hyperbola (d) Straight line √

5- **$\int \sin 3x \, dx$ is:**

(a) $\frac{\cos 3x}{3} + c$ (b) $-\frac{\cos 3x}{3} + c$ √
(c) $3 \cos 3x + c$ (d) $-3 \cos 3x + c$

6- **A function which is to be maximized or minimized is called:**

(a) Subjective function
(b) Quantitative function
(c) Objective function √
(d) Qualitative function

7- $\frac{d}{dx}(a^x)$ is:

(a) $\frac{\ln a}{a^x}$ (b) $\frac{a^x}{\ln a}$

(c) a^x (d) $a^x \ln a$ ✓

8- $\lim\limits_{\theta \to 0} \frac{\sin 7\theta}{\theta}$ is equal to:

(a) $\frac{1}{7}$ (b) 7 ✓

(c) 1 (d) $\frac{2}{7}$

9- $\int_{-1}^{3} x^3\, dx$ is:

(a) 20 ✓ (b) 40

(c) 30 (d) 60

10- The centroid of the triangle whose vertices are (3, –5), (–7, 4) and (10, –2) is:

(a) (–2, –2) (b) (–2, 2)

(c) (2, –1) ✓ (d) (0, 0)

11- $\int \tan x\, dx$ is:

(a) $\ln \sec x + c$ ✓ (b) $\ln \operatorname{cosec} x + c$

(c) $\ln \sin x + c$ (d) $\ln \cot x + c$

12- If $y = \frac{1}{x^2}$, then $\frac{dy}{dx}$ at $x = -1$ is:

(a) 2 ✓ (b) 3

(c) $\frac{1}{3}$ (d) 4

13- $f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \text{------} + \frac{f^n(0)}{n!} + \text{----}$ is called:

(a) Taylor series (b) Binomial series

(c) Laurent series (d) Maclaurin series ✓

14- $\frac{1}{1+x^2}$ is derivative of :

(a) $\sin^{-1} x$
(b) $\sec^{-1} x$
(c) $\tan^{-1} x$ ✓
(d) $\cot^{-1} x$

15- The angle between the vectors $2\bar{i} + 3\bar{j} + \bar{k}$ and $2\bar{i} - \bar{j} - \bar{k}$ is:

(a) 90^o ✓
(b) 45^o
(c) 60^o
(d) 30^o

16- The eccentricity e of the hyperbola is:

(a) $e = 0$
(b) $e < 1$
(c) $e > 1$ ✓
(d) $e = 1$

17- $\hat{j} \times \hat{k}$ is equal to:

(a) $-\hat{i}$
(b) $\hat{i}$ ✓
(c) 0
(d) 1

18- $\frac{d}{dx}(\sec x)$ is:

(a) $\sec^2 x$
(b) $-\text{sex}\, x \tan x$
(c) $\sec x \tan x$ ✓
(d) $\sec x \cot x$

19- $\int_a^b f(x)dx$ is:

(a) $-\int_a^b f(x)dx$
(b) $-\int_b^a f(x)dx$ ✓
(c) $\int_{-b}^{-a} f(x)dx$
(d) $-\int_{-b}^{+a} f(x)dx$

20- If $f(x) = x^2$, then domain of f is:

(a) Real number ✓
(b) Integer
(c) Rational number
(d) Irrational

Inter (Part-II) 2016

Mathematics	Group-II	PAPER: II
Time: 2.30 Hours	(SUBJECTIVE TYPE)	Marks: 80

SECTION-I

2. Write short answers to any EIGHT (8) questions: 16

(i) Prove the identity $\cosh^2 x + \sinh^2 x = \cosh 2x$.

Ans $\cosh^2 x + \sinh^2 x = \frac{e^{2x} + e^{-2x} + 2}{4} + \frac{e^{2x} + e^{-2x} - 2}{4}$

$$= \frac{e^{2x} + e^{-2x} + 2 + e^{2x} + e^{-2x} - 2}{4}$$

$$= \frac{2e^{2x} + 2e^{-2x}}{4}$$

$$= \frac{e^{2x} + e^{-2x}}{2}$$

$\therefore \quad \cosh^2 x + \sinh^2 x = \cosh 2x$

(ii) Evaluate $\lim\limits_{\theta \to 0} \frac{\sin^2 \theta}{\theta}$.

Ans $\lim\limits_{\theta \to 0} \frac{\sin^2 \theta}{\theta} = \lim\limits_{\theta \to 0} \frac{\sin^2 \theta}{\theta^2 \cdot \frac{1}{\theta}}$

$$= \lim_{\theta \to 0} \left[\theta \cdot \left(\frac{\sin \theta}{\theta}\right)^2\right]$$

$$= \lim_{\theta \to 0} \theta \cdot \lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta}\right)^2$$

$$= 0\left[\lim_{\theta \to 0} \frac{\sin \theta}{\theta}\right]^2$$

$$= 0(1)^2 = 0$$

(iii) Discuss the continuity of the function $f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ at $x = 3$.

Ans Given $f(3) = 6$

$\therefore$ the function f is defined at $x = 3$.

Now $\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3}$

$= \lim_{x\to 3} \frac{(x+3)(x-3)}{x-3}$

$= \lim_{x\to 3} (x+3)$

$= 6$

As $\lim_{x\to 3} f(x) = 6 = f(3)$

$\therefore$ f(x) is continuous at x = 3.

(iv) If y = c, find $\frac{dy}{dx}$ by definition where c is constant.

Ans $y = f(x) = c$

$\Rightarrow f(x+\delta x) = c$

$f(x+\delta x) - f(x) = c - c = 0$

$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{0}{\delta x} = 0$

$\lim_{\delta x\to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x\to 0} (0) = 0$

Thus $f'(x) = 0$

That is, $\frac{dy}{dx} = \frac{d}{dx}(c) = 0.$

(v) Find $\frac{dy}{dx}$ if $y = \frac{a+x}{a-x}$.

Ans $\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{a+x}{a-x}\right)$

$= \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$

$= \frac{(a-x)(0+1) - (a+x)(0-1)}{(a-x)^2}$

$= \frac{a-x+a+x}{(a-x)^2}$

$= \frac{2a}{(a-x)^2}$

(vi) Find $\frac{dy}{dx}$ if $xy + y^2 = 2$.

Ans $xy + y^2 = 2$

Differentiating the equation w.r.t. x:

$$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(2)$$

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$

$$x\frac{d}{dx}(y) + y\frac{d}{dx}(x) + 2y\frac{d}{dx}(y) = 0$$

$$x\frac{dy}{dx} + 2y\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx}(x + 2y) = -y$$

$$\frac{dy}{dx} = -\frac{y}{x + 2y}$$

(vii) Find $\frac{dy}{d\theta}$ if $y = (\sin 2\theta - \cos 3\theta)^2$.

Ans $y = (\sin 2\theta - \cos 3\theta)^2$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}[(\sin 2\theta - \cos \theta)^2]$$

$$= 2(\sin 2\theta - \cos 3\theta)\frac{d}{d\theta}(\sin 2\theta - \cos 3\theta)$$

$$= 2(\sin 2\theta - \cos 3\theta)[\cos 2\theta . 2 - (-\sin 3\theta . 3)]$$

$$= 2(\sin 2\theta - \cos 3\theta)(2\cos 2\theta + 3\sin 3\theta)$$

(viii) Differentiate $y = a^x$ w.r.t. x.

Ans Here

$$y = a^x$$

Taking log on both sides

$$\log y = \log a^x$$

$$\ln y = x \ln a$$

$$= e^{x \ln a}$$

Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = e^{x \ln a}\frac{d}{dx}(x \ln a)$$

$$= e^{x \ln a}(\ln a)$$

$$= a^x(\ln a) \qquad (\because e^{x \ln a} = a^x)$$

(ix) If $y = \cos h\, x$ prove that $\frac{dy}{dx} = \sin h\, x$.

Ans

$$y = \cos h\, x$$

$$\frac{dy}{dx} = \frac{d}{dx}(\cos h\, x)$$

$$= \frac{d}{dx}\left[\frac{1}{2}(e^x + e^{-x})\right]$$

$$= \frac{1}{2}[e^x + e^{-x}.(-1)]$$

$$= \frac{1}{2}(e^x - e^{-x})$$

$$= \sin hx$$

(x) Find $\frac{dy}{dx}$ if $y = x\, e^{\sin x}$.

Ans

$$y = x\, e^{\sin x}$$

$$\frac{dy}{dx} = \frac{d}{dx}(x\, e^{\sin x})$$

$$= x\frac{d}{dx}(e^{\sin x}) + e^{\sin x}\frac{d}{dx}(x)$$

$$= x\,.\,e^{\sin x}\,.\cos x + e^{\sin x}\,.\,1$$

$$= e^{\sin x}(x \cos x + 1)$$

(xi) State Taylor's series of a function f(x) at x = a.

Ans If f is defined in the interval containing 'a' and its derivatives of all orders exist at x = a, then we can expand f(x) as:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{\lfloor 2}(x - a)^2 + \frac{f'''(a)}{\lfloor 3}(x - a)^3 + \frac{f^{(4)}(a)}{\lfloor 4}(x - a)^4 + \ldots + \frac{f^{(n)}(a)}{\lfloor n}(x - a)^n + \ldots$$

(xii) If $f(x) = x^3 - 6x^2 + 9x$, determine the interval in which f(x) is decreasing.

Ans $f(x) = x^3 - 6x^2 + 9x$

$$f'(x) = 3x^2 - 12x + 9$$

$$= 3(x^2 - 4x + 3)$$

$$= 3(x - 1)(x - 3)$$

$$f'(x) > 0$$

$$\Rightarrow \quad x^2 - 4x + 3 > 0$$

$$\Rightarrow \quad (x - 1)(x - 3) > 0$$

$(x - 1)(x - 3) > 0$ in the interval $(-\infty, 1)$ and $(3, \infty)$

$f'(x) < 0 \Rightarrow (x - 1)(x - 3) < 0$

$(x - 1)(x - 3) < 0$ if $x > 1$ and $x < 3$, that is $1 < x < 3$.

3. **Write short answers to any EIGHT (8) questions: 16**

(i) If $xy + x = 4$, find $\frac{dx}{dy}$ by using differentials.

Ans $xy + x = 4$

Taking differential on both sides

$$d(xy + x) = d(4)$$

$$x\,dy + y\,dx + dx = 0 \quad (1)$$

$$x\,dy + dx(y + 1) = 0$$

$$x\,dy = -dx\,(y + 1)$$

$$\frac{dy}{dx} = -\frac{(y + 1)}{x}$$

From (1)

$$dx\,(y + 1) = -x\,dy$$

$$\frac{dx}{dy} = -\frac{x}{(y + 1)}$$

So,

$$\frac{dy}{dx} = -\frac{(y + 1)}{x}$$

and

$$\frac{dx}{dy} = -\frac{x}{(y + 1)}$$

(ii) Evaluate $\int \frac{1}{\sqrt{x+1} - \sqrt{x}}\,dx$.

Ans Rationalizing the denominator, we have

$$\int \frac{dx}{\sqrt{x+1} - \sqrt{x}} = \int \frac{\sqrt{x+1} + \sqrt{x}}{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}\,dx$$

$$= \int \frac{\sqrt{x+1} + \sqrt{x}}{x + 1 - x}\,dx$$

$$= \int [(x+1)^{1/2} + x^{1/2}]\,dx$$

$$= \int (x + 1)^{1/2}\,dx + \int x^{1/2}\,dx$$

$$= \frac{(x + 1)^{3/2}}{\frac{3}{2}} + \frac{x^{3/2}}{\frac{3}{2}} + c$$

$$= \frac{2}{3}(x + 1)^{3/2} + \frac{2}{3}x^{3/2} + c$$

(iii) Evaluate $\int \frac{ax + b}{ax^2 + 2bx + c}$.

Ans $\int \frac{ax+b}{ax^2+2bx+c}\,dx = \frac{1}{2}\int \frac{2ax+2b}{ax^2+2bx+c}\,dx$

$$= \frac{1}{2}\ln|ax^2+2bx+c| + c'$$

(iv) **Evaluate $\int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx$.**

Ans Let $\tan x = t$

$\therefore \sec^2 x\,dx = dt$

$$\int \frac{\sec^2 x\,dx}{\sqrt{\tan x}} = \int \frac{dt}{t^{1/2}}$$

$$= \int t^{-1/2}\,dt$$

$$= \frac{t^{1/2}}{\frac{1}{2}} + c$$

$$= 2(t^{1/2}) + c$$

$$= 2(\tan x)^{1/2} + c$$

$$= 2\sqrt{\tan x} + c$$

(v) **Evaluate $\int \tan^{-1} x\,dx$.**

Ans $\int \tan^{-1} x\,dx = \int \tan^{-1} x \,.\, 1\,dx$

Integrating by parts:

$$\int \tan^{-1} x\,dx = \tan^{-1} x \int dx - \int\left[\frac{d}{dx}(\tan^{-1} x)\,.\int dx\right]dx$$

$$= \tan^{-1} x\,.\,x - \int\left[\frac{1}{1+x^2}\,.\,x\right]dx$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$

$$= x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + c$$

(vi) **Evaluate $\int_2^{\sqrt{5}} x\sqrt{x^2-1}\,dx$.**

Ans $\int_2^{\sqrt{5}} x\sqrt{x^2-1}\,dx = \int_2^{\sqrt{5}} x(x^2-1)^{1/2}\,dx$

$$= \frac{1}{2}\int_2^{\sqrt{5}} 2x(x^2-1)^{1/2}\,dx$$

$$= \frac{1}{2}\left[\frac{(x^2-1)^{3/2}}{\frac{3}{2}}\right]_2^{\sqrt{5}}$$

$$= \frac{1}{2}\left(\frac{2}{3}\right)\left[(x^2-1)^{3/2}\right]_2^{\sqrt{5}}$$

$$= \frac{1}{3}[(5-1)^{3/2} - (4-1)^{3/2}]$$

$$= \frac{1}{3}[((4)^{1/2})^3 - ((3)^3)^{1/2}]$$

$$= \frac{1}{3}[2^3 - (27)^{1/2}]$$

$$= \frac{1}{3}[8 - 3\sqrt{3}]$$

(vii) Evaluate $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx$.

Ans Let $\tan^{-1} x = t$

By taking derivative both sides

$$\left(\frac{1}{1+x^2}\right) dx = dt$$

$$\int \frac{e^{\tan^{-1} x}}{1+x^2} dx = \int e^t dt$$

$$= \frac{e^t}{1} + c$$

$$= e^t + c$$

$$= e^{\tan^{-1} x} + c$$

(viii) Evaluate $\int x^2 \ln x \, dx$.

Ans $\int x^2 \ln x \, dx = \int \ln x \, . \, x^2 \, dx$

Integrating by parts

$$= \ln x \int x^2 dx - \int\left[\frac{d}{dx}(\ln x) \, . \int x^2 dx\right] dx$$

$$= \ln x \frac{x^3}{3} - \int\left[\frac{1}{x} \cdot \frac{x^3}{3}\right] dx$$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2 dx$$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3} \cdot \frac{x^3}{3} + c$$

$$= \frac{x^3}{3} (\ln x - \frac{1}{3}) + c$$

(ix) **Solve $x^2(2y + 1) \frac{dy}{dx} - 1 = 0$.**

Ans The given differential equation can be written as:

$$x^2(2y + 1) \frac{dy}{dx} = 1 \qquad (1)$$

Dividing by x^2, we have $(2y + 1) \frac{dy}{dx} = \frac{1}{x^2}$ (2)

Multiplying both sides of (1) by dx, we get

$$(2y + 1) \left(\frac{dy}{dx}\right) dx = \frac{1}{x^2} dx$$

$$(2y + 1)\, dy = \frac{1}{x^2} dx$$

Integrating either sides, we get

$$\int (2y + 1)\, dy = \int \frac{1}{x^2} dx$$

$$y^2 + y = \frac{-1}{x} + c$$

Thus $y^2 + y = c - \frac{1}{x}$ is the general solution of the given differential equation.

(x) **Show that $y = \tan(e^x + c)$ is solution of $\frac{dy}{dx} = \frac{y^2 + 1}{e^{-x}}$.**

Ans

$$y = \tan(e^x + c)$$

$$\frac{dy}{dx} = \frac{d}{dx}[\tan(e^x + c)]$$

$$\frac{dy}{dx} = \frac{y^2 + 1}{e^{-x}}$$

$$\frac{dy}{y^2 + 1} = \frac{dx}{e^{-x}}$$

$$\int \frac{dy}{1 + y^2} = \int e^x + dx$$

$$\tan^{-1} y = e^x + c$$

$$y = \tan(e^x + c) \qquad \text{True}$$

(xi) Graph $x + y \leq 5, -2x + y \leq 2, y \geq 0$.

Ans

$$x + y \leq 5 \quad \text{(i)}$$
$$-2x + y \leq 2 \quad \text{(ii)}$$
$$x + y = 5 \quad \text{(iii)}$$
$$-2x + y = 2 \quad \text{(iv)}$$

Putting $x = 0$ in (iii):

$$0 + y = 5$$
$$\Rightarrow \quad y = 5$$

$\therefore$ (0, 5) is a point on (iii).

Putting $y = 0$ in (iii)

$$x + 0 = 5$$
$$\Rightarrow \quad x = 5$$

$\therefore$ (5, 0) is another point on (iii).

$$-2x + y \leq 2 \quad \text{(ii)}$$
$$-2x + y = 2 \quad \text{(iv)}$$

Putting $x = 0$ in (iv)

$$-2x + 0 = 2$$
$$\Rightarrow \quad x = -1$$

$\therefore$ (–1, 0) is another point on (iv).

Putting $x = 0, y = 0$ in (i), we get

$$0 + 0 < 5$$

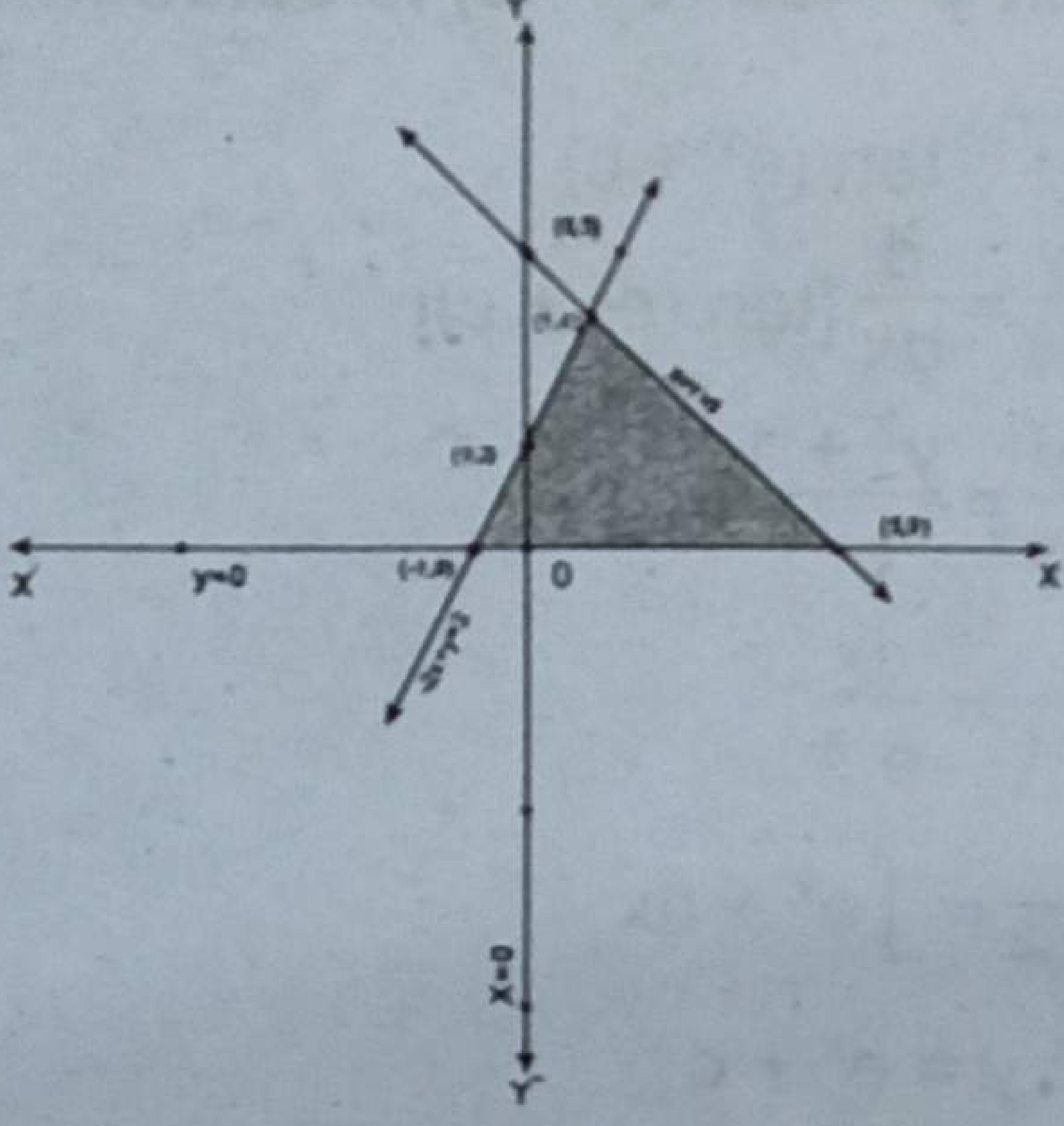
$\therefore \quad 0 < 5$ True

Putting $x = 0, y = 0$ in (ii).

$$-0 + 0 < 2$$

$\therefore \quad 0 < 2$ True

Graph:



(xii) **Find corner points of $5x + 7y \leq 35$, $x - 2y \leq 4$, $x \geq 0$, $y \geq 0$.**

Ans

$$5x + 7y \leq 35 \quad \text{(i)}$$
$$x - 2y \leq 4 \quad \text{(ii)}$$
$$5x + 7y = 35 \quad \text{(iii)}$$
$$x - 2y = 4 \quad \text{(iv)}$$

Putting $x = 0$ in (iii)

$$0 + 7y + 35 \Rightarrow y = 5$$

$\therefore$ (0, 5) is a point on (iii).

Putting $y = 0$ in (iii)

$$5x + 0 = 35$$
$$x = 7$$

$\therefore$ (7, 0) is another point on (iii).

Putting $x = 0$ in (iv)

$$0 - 2y = 4$$
$$y = -2$$

$\therefore$ (0, –2) is a point on (iv).

Putting $y = 0$ in (iv)

$\therefore$ (4, 0) is another point on (iv).

Putting $x = 0, y = 0$ in (i)

$$0 + 0 < 35$$

$\therefore$ $0 < 35$ True

Putting $x = 0, y = 0$ in (ii)

$$0 - 0 < 4$$

$\therefore$ $0 < 4$ True

From (iv)

$$x = 4 + 2y \quad \text{(v)}$$

Putting (v) in (iii)

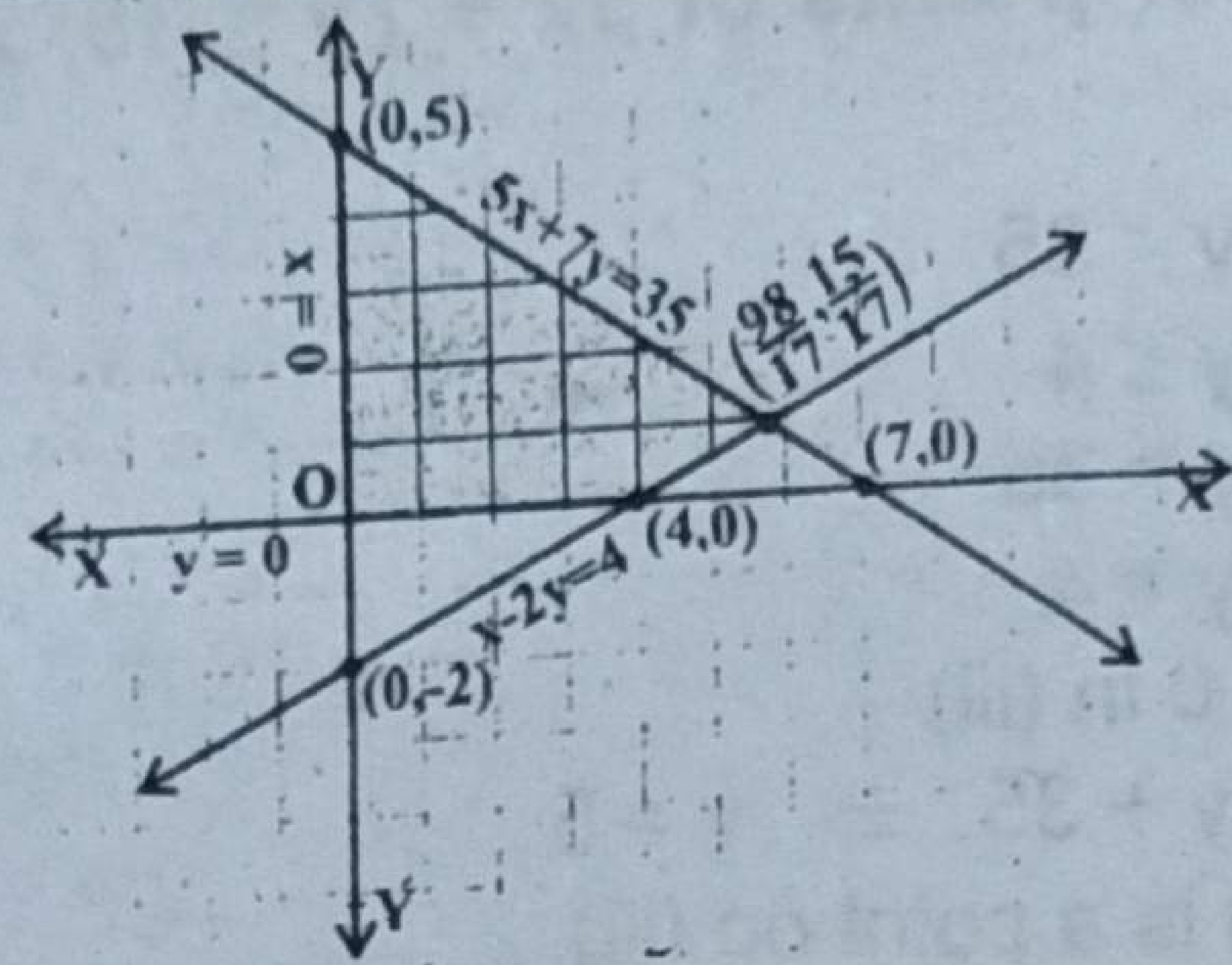
$$5(4 + 2y) + 7y = 35$$
$$20 + 10y + 7y = 35$$
$$17y = 15$$

$\therefore$ $y = \frac{15}{17}$

Putting $y = \frac{15}{17}$ in (v)

$$x = 4 + 2\left(\frac{15}{17}\right) = 4 + \frac{30}{17} = \frac{68 + 30}{17} = \frac{98}{17}$$

Hence $\left(\frac{98}{17}, \frac{15}{17}\right)$, (0, 5), (0, 0) and (4, 0) are the corner points.



4. Write short answers to any NINE (9) questions: 18

(i) Find the equation of the line through A(–6, 5) having slope 7.

Ans $m = 7$

Equation of the line through (–6, 5) is:

$$y - y_1 = m(x - x_1)$$
$$y - 5 = 7(x + 6)$$
$$y - 5 = 7x + 42$$
$$y = 7x + 42 + 5$$
$$= 7x + 47$$
$$7x - y + 47 = 0$$

(ii) Convert the equation $4x + 7y - 2 = 0$ into two intercepts form.

Ans $4x + 7y - 2 = 0$

$$4x + 7y = 2$$
$$\frac{4}{2}x + \frac{7}{2}y = \frac{2}{2}$$
$$2x + \frac{7}{2}y = 1$$
$$\frac{x}{\frac{1}{2}} + \frac{y}{\frac{2}{7}} = 1$$

(iii) Check whether the point (–2, 4) lies above or below the line $4x + 5y - 3 = 0$.

Ans $4x + 5y - 3 = 0$ (1)

Here b = 5 is positive. Also

$$4(-2) + 5(4) - 3 = -8 + 20 - 3$$
$$= 9 > 0$$ (2)

The coefficient of y in (1) and the expression have the same sign and so the point (–2, 4) lies above (1).

(iv) **Find the lines represented by $20x^2 + 17xy - 24y^2 = 0$.**

Ans $20x^2 + 17xy - 24y^2 = 0$

This equation may be written as:

$$24\left(\frac{y}{x}\right)^2 - 17\left(\frac{y}{x}\right) - 20 = 0$$

$$a = 24 \quad , \; b = -17 \quad , \; c = -20$$

$$= \frac{-(-17) \pm \sqrt{(-17)^2 - 4(24)(-20)}}{2(24)}$$

$$\Rightarrow \quad \frac{y}{x} = \frac{17 \pm \sqrt{289 + 1920}}{48}$$

$$\frac{y}{x} = \frac{17 + 47}{48} \quad ; \quad \frac{17 - 47}{48}$$

$$= \frac{4}{3}, \frac{-5}{8}$$

$$\Rightarrow \quad y = \frac{4}{3}x \quad \text{and} \quad y = \frac{-5}{8}x$$

$$\Rightarrow \quad 4x - 3y = 0 \quad \text{and} \quad 5x + 8y = 0$$

(v) **Find an equation of the vertical line through (–5, 3).**

Ans Slope = $m = \tan 90^\circ = \infty$

Equation of line through (–5, 3) is

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \infty\,(x + 5)$$

$$\frac{y - 3}{x + 5} = \infty$$

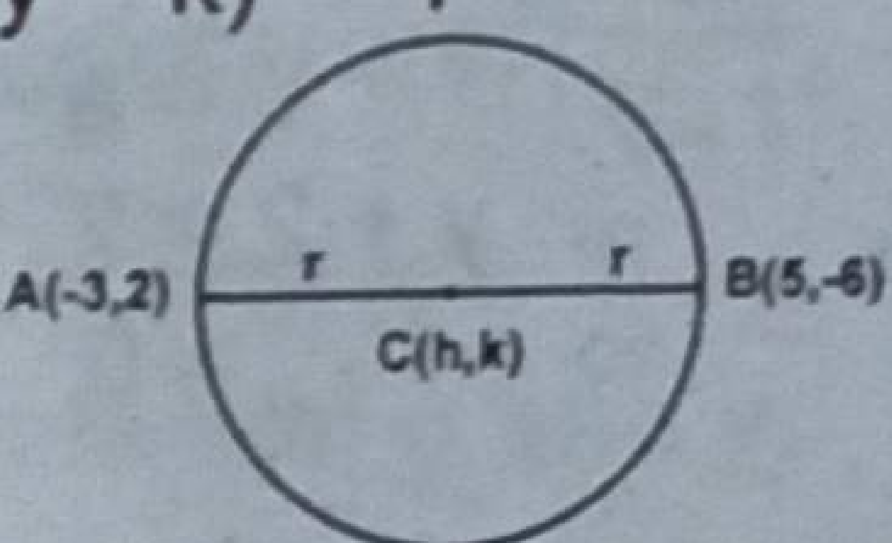
$$\Rightarrow \quad x + 5 = 0$$

$$\therefore \quad x = -5$$

(vi) **Find an equation of the circle with ends of diameter at (–3, 2) and (5, –6).**

Ans Let the equation of required circle is:

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \text{(i)}$$



Since C is the centre of circle, thus

$h = \frac{-3+5}{2}$; $k = \frac{2+(-6)}{2}$

$= \frac{2}{2}$; $= \frac{-4}{2}$

$= 1$; $= -2$

$\therefore \quad C(h, k) = C(1, -2)$

By distance formula

$$r = |AC| = \sqrt{(1+3)^2 + (-2-2)^2}$$
$$= \sqrt{16+16}$$
$$= \sqrt{32}$$

Putting the values in (i)

$$(x-1)^2 + (y+2)^2 = (\sqrt{32})^2$$
$$x^2 - 2x + 1 + y^2 + 4y + 4 = 32$$
$$x^2 + y^2 - 2x + 4y - 27 = 0$$

(vii) Find the centre and radius of the circle $x^2 + y^2 + 12x - 10y = 0$.

Ans $x^2 + y^2 + 12x - 10y = 0$

$$x^2 + y^2 + 2(6)x + 2(-5)y + 0 = 0$$

$\therefore \quad g = 6, f = -5, c = 0$

So, Centre $(-g, -f)$ = Centre $(-6, 5)$

and Radius $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{(6)^2 + (-5)^2 - 0}$$
$$= \sqrt{36+25}$$
$$= \sqrt{61}$$

(viii) Find the focus and vertex of parabola $(x-1)^2 = 8(y+2)$.

Ans $(x-1)^2 = 8(y+2)$

Let $X = x - 1, Y = y + 2$

So, $X^2 = 8Y$ (i)

Comparing (i) with $x^2 = 4ay$

$4a = 8$

$\Rightarrow \quad a = 2$

Coordinates of focus: $(0, a)$

$\therefore \quad X = 0$, ; $Y = a$

$\therefore \quad x - 1 = 0$; $y + 2 = 2$

$\Rightarrow \quad x = 1$; $\Rightarrow \quad y = 0$

$\therefore \quad F(1, 0)$

Coordinates of vertex: $(0, 0)$

$\therefore \quad X = 0, \quad ; \quad Y = 0$

$x - 1 = 0 \quad ; \quad y + 2 = 0$

$x = 1 \quad ; \quad y = -2$

Thus, $V(1, -2)$

(ix) **Find the vertices and directrices of the ellipse $25x^2 + 9y^2 = 225$.**

Ans $25x^2 + 9y^2 = 225$

$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

$\therefore \quad a^2 = 25$

$\Rightarrow \quad a = \pm 5$

$\therefore \quad b^2 = 9$

$\Rightarrow \quad b = \pm 3$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{25 - 9}{25} = \frac{16}{25}$$

$$\Rightarrow \quad e = \frac{4}{5}$$

$$ae = 5\left(\frac{4}{5}\right) = 4$$

$$\frac{a}{e} = \frac{5}{\frac{4}{5}} = 5 \times \frac{5}{4} = \frac{25}{4}$$

Vertices are:

$V(0, \pm a) = V(0, \pm 5)$

Directries:

$$y = \pm\frac{a}{e} = \pm\frac{25}{4}$$

(x) **Find the direction cosines of the vector $6\underline{i} - 2\underline{j} + \underline{k}$.**

Ans

$$\underline{v} = 6\underline{i} - 2\underline{j} + \underline{k}$$

$$|\underline{v}| = \sqrt{(6)^2 + (-2)^2 + (1)^2}$$

$$= \sqrt{36 + 4 + 1} = \sqrt{41}$$

Direction rations of $\underline{v}$ are: $(6, -2, 1)$

Direction cosines of $\underline{v}$ are: $\left(\frac{6}{\sqrt{41}}, \frac{-2}{\sqrt{41}}, \frac{1}{\sqrt{41}}\right)$

(xi) **If $\underline{u} = 3\underline{i} + \underline{j} - \underline{k}$ and $\underline{v} = 3\underline{i} - \underline{j} + \underline{k}$, find the cosines of the angle θ between $\underline{u}$ and $\underline{v}$.**

Ans

$$\underline{u} = 3\underline{i} + \underline{j} - \underline{k}$$
$$\underline{v} = 2\underline{i} - \underline{j} + \underline{k}$$
$$\underline{u} \cdot \underline{v} = (3)(2) + (1)(-1) + (-1)(1)$$
$$= 6 - 1 - 1 = 4$$
$$|\underline{u}| = \sqrt{(3)^2 + (1)^2 + (-1)^2} = \sqrt{9+1+1} = \sqrt{11}$$
$$|\underline{v}| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{4+1+1} = \sqrt{6}$$
$$\cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| \cdot |\underline{v}|}$$
$$= \frac{4}{\sqrt{11}\sqrt{6}}$$
$$= \frac{4}{\sqrt{66}}$$

(xii) If $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = \underline{i} - \underline{j} + \underline{k}$, find the cross product $\underline{a} \times \underline{b}$.

Ans

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$
$$= \underline{i}(1-1) - \underline{j}(2+1) + \underline{k}(-2-1)$$
$$= 0\underline{i} - 3\underline{j} - 3\underline{k}$$

(xiii) Find the value of $2\underline{i} \times 2\underline{j} \cdot \underline{k}$.

Ans $2\underline{i} \times 2\underline{j} \cdot \underline{k}$.

$$= [(2\underline{i} + 0\underline{j} + 0\underline{k}) \times (0\underline{i} + 2\underline{j} + 0\underline{k})] \cdot (0\underline{i} + 0\underline{j} + \underline{k})$$
$$= \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= 2(2-0) - 0 + 0$$
$$= 4$$

SECTION-II

NOTE: Attempt any Three (3) questions.

Q.5.(a) Let $f(x) = \dfrac{2x+1}{x-1}$; $x \neq 1$, find $f^{-1}(x)$ and verify that $fof^{-1}(x) = x$. (5)

Ans Let $y = f(x) = \dfrac{2x+1}{x-1}$

$\therefore \quad y(x-1) = 2x+1$

$$xy - y = 2x + 1$$
$$x - 2x = y + 1$$
$$x(y - 2) = y + 1$$
$$x = \frac{y + 1}{y - 2}$$
$$\therefore \quad x = f^{-1}(y)$$

Therefore

$$f^{-1}(y) = \frac{y + 1}{y - 2}$$

Hence $f^{-1}(x) = \frac{x + 1}{x - 2}$

Verification:

$$fof^{-1}(x) = x$$
$$fof^{-1}(x) = f[f^{-1}(x)]$$
$$= \frac{2\left(\frac{x + 1}{x - 2}\right) + 1}{\frac{x + 1}{x - 2} - 1}$$
$$= \frac{\frac{2x + 2 + x - 2}{x - 2}}{\frac{x + 1 - x + 2}{x - 2}}$$
$$= \frac{3x}{3}$$
$$fof^{-1}(x) = x \qquad \text{Proved.}$$

(b) If $\frac{y}{x} = \tan^{-1}\left(\frac{x}{y}\right)$, then prove that $\frac{dy}{dx} = \frac{y}{x}$. **(5)**

Ans

$$\frac{y}{x} = \tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating w.r.t. 'x'

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$\frac{x\frac{d}{dx}(y) - y\frac{d}{dx}(x)}{x^2} = \frac{1}{1 + \left(\frac{x}{y}\right)^2}\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\frac{x\frac{dy}{dx}-y}{x^2}=\frac{1}{\frac{y^2+x^2}{y^2}}\left[\frac{y\frac{d}{dx}(x)-x\frac{d}{dx}(y)}{y^2}\right]$$

$$=\frac{y^2}{x^2+y^2}\left[\frac{y-x\frac{dy}{dx}}{y^2}\right]$$

$$(x^2+y^2)\left(x\frac{dy}{dx}-y\right)=x^2\left[y-x\frac{dy}{dx}\right]$$

$$x^3\frac{dy}{dx}-x^2y+xy^2\frac{dy}{dx}-y^3=x^2y-x^3\frac{dy}{dx}$$

$$2x^3\frac{dy}{dx}+xy^2\frac{dy}{dx}=2x^2y+y^3$$

$$\frac{dy}{dx}(2x^3+xy^2)=2x^2y+y^3$$

$$\frac{dy}{dx}=\frac{y(2x^2+y^2)}{x(2x^2+y^2)}$$

$$\frac{dy}{dx}=\frac{y}{x}$$

Q.6.(a) Evaluate $\int\left(\frac{1-\sin x}{1-\cos x}\right)e^x\,dx$. (5)

Ans $\int\left(\frac{1-\sin x}{1-\cos x}\right)e^x\,dx$

$$=\int e^x\left[\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right]dx$$

$$=\int e^x\left[\frac{1}{2\sin^2\frac{x}{2}}-\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right]dx$$

$$=\int e^x\left[\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}-\cot\frac{x}{2}\right]dx$$

$$=\frac{1}{2}\int e^x\operatorname{cosec}^2\frac{x}{2}\,dx-\int e^x\cot\frac{x}{2}\,dx$$

Integrating 1^{st} integral by parts

$$=\frac{1}{2}e^x\int\operatorname{cosec}^2\frac{x}{2}\,dx-\frac{1}{2}\int\left[\frac{d}{dx}(e^x)\int\operatorname{cosec}^2\frac{x}{2}\,dx\right]dx$$

$$- \int e^x \cot\frac{x}{2}\,dx + c$$

$$= \frac{1}{2}e^x \cdot \left(\frac{-\cot\frac{x}{2}}{\frac{1}{2}}\right) - \frac{1}{2}\int e^x \left(\frac{-\cot\frac{x}{2}}{\frac{1}{2}}\right) dx - \int e^x \cot\frac{x}{2}\,dx + c$$

$$= -e^x \cot\frac{x}{2} + \int e^x \cot\frac{x}{2}\,dx - \int e^x \cot\frac{x}{2}\,dx + c$$

$$= -e^x \cot\frac{x}{2} + c$$

(b) Find interior angles of a triangle whose vertices are A(6, 1), B(2, 7), C(–6, –7). **(5)**

Ans A(6, 1), B(2, 7), C(–6, –7)

$$\text{Slope of AB} = m_1 = \frac{7-1}{2-6} \Rightarrow \frac{6}{-4} = \frac{-3}{2}$$

$$\text{Slope of BC} = m_2 = \frac{-7-7}{-6-2} \Rightarrow \frac{-14}{-8} = \frac{7}{4}$$

$$\text{Slope of AC} = m_3 = \frac{-7-1}{-6-6} \Rightarrow \frac{-8}{-12} = \frac{2}{3}$$

$$\tan(\angle A) = \frac{m_3 - m_1}{1 + m_3 \cdot m_1}$$

$$= \frac{\frac{2}{3} + \frac{3}{2}}{1 + \left(\frac{2}{3}\right)\left(\frac{-3}{2}\right)} = \frac{\frac{13}{6}}{0} = \infty$$

$$\angle A = \tan^{-1}(\infty) = 90^\circ$$

$$\tan(\angle B) = \frac{m_1 - m_2}{1 + m_1 \cdot m_2} = \frac{\frac{-3}{2} - \frac{7}{4}}{1 + \left(\frac{-3}{2}\right)\left(\frac{7}{4}\right)}$$

$$= \frac{\frac{-13}{4}}{\frac{-13}{8}} = 2$$

$$\angle B = \tan^{-1} 2 = 63.43^\circ$$

$$\tan(\angle C) = \frac{m_2 - m_3}{1 + m_2 . m_3} = \frac{\frac{7}{4} - \frac{2}{3}}{1 + \left(\frac{7}{4}\right)\left(\frac{2}{3}\right)} = \frac{\frac{13}{12}}{\frac{26}{12}}$$

$$= \frac{1}{2}$$

$$\angle C \tan^{-1}\left(\frac{1}{2}\right) = 26.57$$

Q.7.(a) Evaluate $\int_0^{\pi/4} \frac{\sec\theta}{\sin\theta + \cos\theta} d\theta$. (5)

Ans

$$\int_0^{\pi/4} \frac{\sec\theta}{\sin\theta + \cos\theta} d\theta$$

$$= \int_0^{\pi/4} \frac{\frac{\sec\theta}{\cos\theta}}{\frac{\sin\theta + \cos\theta}{\cos\theta}} d\theta = \int_0^{\pi/4} \frac{\sec\theta . \frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\cos\theta}} d\theta$$

$$= \int_0^{\pi/4} \frac{\sec^2\theta}{1 + \tan\theta} d\theta$$

$$= \ln|1 + \tan\theta|]_0^{\pi/4}$$

$$= \ln|1 + \tan\frac{\pi}{4}| - \ln|1 + \tan(0)|$$

$$= \ln|1 + 1| - \ln|1 + 0|$$

$$= \ln(2) - 0$$

$$= \ln(2)$$

(b) Maximize $f(x, y) = 2x + 3y$ subject to the constraints: (5)
$2x + y \le 8, x + 2y \le 14, x \ge 0, y \ge 0$.

Ans

$2x + y \le 8$ (i)

$x + 2y \le 14$ (ii)

$2x + y = 8$ (iii)

$x + 2y = 14$ (iv)

Putting $x = 0$ in (iii)

$0 + y = 8$

$\Rightarrow y = 8$

$\therefore$ (0, 8) is a point on (iii).

Putting $y = 0$ in (iii).

$$2x + 0 = 8$$
$$\Rightarrow \quad x = 4$$

$\therefore$ (4, 0) is another point on (iii).

Putting $x = 0$ in (iv)

$$0 + 2y = 14$$
$$\Rightarrow \quad y = 7$$

$\therefore$ (0, 7) is a point on (iv).

Putting $y = 0$ in (iv)

$$x + 0 = 14$$
$$\Rightarrow \quad x = 14$$

$\therefore$ (14, 0) is another point on (iv).

Putting $x = 0, y = 0$ in (i).

$$0 + 0 < 8$$

$\therefore \quad 0 < 8$ True (Maximized)

Putting $x = 0, y = 0$ in (ii).

$$0 + 0 < 14$$

$\therefore \quad 0 < 14$ True (Maximized)

Q.8.(a) Show that $3x - 2y = 0$ is tangent to the circle $x^2 + y^2 + 6x - 4y = 0$. **(5)**

Ans $x^2 + y^2 + 6x - 4y = 0$

$$x^2 + y^2 + 2(3)x + 2(-2)y + 0 = 0$$

$\therefore \quad g = 3, f = -2, c = 0$

$\therefore \quad$ centre $= c'(-g, -f) = c'(-3, 2)$

and radius $= r = \sqrt{g^2 + f^2 - c}$

$$= \sqrt{(3)^2 + (-2) - 0}$$
$$= \sqrt{9 + 4} = \sqrt{13}$$

Perpendicular distance between the centre and the line $3x - 2y = 0$ is the radius of the circle, *i.e.*,

$$r = \frac{|(3)(-3) + (-2)(2) + (0)|}{\sqrt{(3)^2 + (-2)^2}}$$
$$= \frac{|-9 - 4|}{\sqrt{9 + 4}} = \frac{13}{\sqrt{13}} = \sqrt{13}$$

(b) If $\underline{a} = 3\underline{i} - \underline{j} - 4\underline{k}$; $\underline{b} = -2\underline{i} - 4\underline{j} - 3\underline{k}$ and $\underline{c} = \underline{i} + 2\underline{j} - \underline{k}$, then find a unit vector parallel to $3\underline{a} - 2\underline{b} + 4\underline{c}$. **(5)**

Ans Let $\underline{v} = 3\underline{a} - 2\underline{b} + 4\underline{c}$

By putting the values of $\underline{a}$, $\underline{b}$ and $\underline{c}$

$$\underline{v} = 3(3\underline{i} - \underline{j} - 4\underline{k}) - 2(-2\underline{i} - 4\underline{j} - 3\underline{k}) + 4(\underline{i} + 2\underline{j} - \underline{k})$$
$$= 9\underline{i} - 3\underline{j} - 12\underline{k} + 4\underline{i} + 8\underline{j} + 6\underline{k} + 4\underline{i} + 8\underline{j} - 4\underline{k}$$
$$= 17\underline{i} + 13\underline{j} - 10\underline{k}$$
$$|\underline{v}| = \sqrt{(17)^2 + (13)^2 + (-10)^2}$$
$$= \sqrt{289 + 169 + 100}$$
$$= \sqrt{558}$$
$$\hat{v} = \frac{v}{|\underline{v}|} = \frac{17\underline{i} + 13\underline{j} - 10\underline{k}}{\sqrt{558}}$$
$$= \frac{17}{\sqrt{558}}\underline{i} + \frac{13}{\sqrt{558}}\underline{j} - \frac{10}{\sqrt{558}}\underline{k}$$

Q.9.(a) Write an equation of the parabola with axis $y = 0$ and passing through (2, 1) and (11, –2). (5)

Ans General equation of required parabola:

$$(y - k)^2 = 4a(x - h)$$

Axis : $y = 0$

Therefore, $k = 0$

Hence

$$y^2 = 4a(x - h) \quad \text{(i)}$$

$\therefore$ the points (2, 1) and (11, –2) lies on it, thus, put them in (i).

$$(1)^2 = 4a(2 - h)$$
$$1 = 8a - 4ah \quad \text{(ii)}$$

and

$$(-2)^2 = 4a(11 - h)$$
$$4 = 44a - 4ah \quad \text{(iii)}$$

By subtracting (ii) and (iii), we get

$$1 = 8a - 4ah$$
$$\pm 4 = \underset{-}{44a} \underset{+}{-} 4ah$$
$$-3 = -36a$$
$$\Rightarrow \quad a = \frac{1}{12}$$

Put the value of 'a' in (ii)

$$1 = 8\left(\frac{1}{12}\right) - 4\left(\frac{1}{12}\right)h$$

$$1 = \frac{2}{3} - \frac{h}{3}$$

$\Rightarrow \quad 3(1) = 2 - h$

$\Rightarrow \quad h = -1$

By putting the values of a and h in (i), we get

$$y^2 = 4\left(\frac{1}{12}\right)(x + 1)$$

$$y^2 = \frac{x + 1}{3}$$

$\therefore \quad 3y^2 = x + 1$

(b) Find volume of the tetrahedron with the vertices: (5)
(2, 1, 8), (3, 2, 9), (2, 1, 4) and (3, 3, 10)

Ans For Answer see Paper 2016 (Group-I), Q.9.(b).