Inter (Part-II) 2019

Mathematics	Group-II	PAPER: II
Time: 30 Minutes	(OBJECTIVE TYPE)	Marks: 20

Note: Four possible answers A, B, C and D to each question are given. The choice which you think is correct, fill that circle in front of that question with Marker or Pen ink in the answer-book. Cutting or filling two or more circles will result in zero mark in that question.

1-1- $\frac{d}{dx}(\sqrt{x}) =$:

(a) $\sqrt{x}$ (b) $\frac{1}{\sqrt{x}}$

(c) $\frac{1}{2x}$ (d) $\frac{1}{2\sqrt{x}}$ ✓

2- $\int \tan x \, dx =$:

(a) $\ln|\sec x| + c$ ✓ (b) $\ln|\cos ecx| + c$

(c) $\ln|\sin x| + c$ (d) $\ln|\cot x| + c$

3- $\int \frac{e^x}{e^x + 3} dx =$:

(a) $\ln(e^x + 3) + c$ ✓ (b) $e^{2x} + c$

(c) $e^0 + c$ (d) $e^{2x} + 3 + c$

4- $\frac{d}{dx}(\cos x^2) =$:

(a) $2x \sin x^2$ (b) $-2x \sin x^2$ ✓

(c) $2 \cos x$ (d) $-2 \sin x$

5- If $y = \sin^{-1}\frac{x}{a}$, then $\sin y =$:

(a) $\cos y$ (b) $\cos x$

(c) $\frac{x}{a}$ ✓ (d) $\frac{y}{a}$

6- The function $y = 27 + x^2$ is a / an:

(a) Constant function (b) Even function

(c) Implicit function (d) Explicit function ✓

7- A function $f(x)$ has relative maximum at $x = c$, if $f'(c) = 0$ and :

(a) $f''(c) > 0$ (b) $f''(c) < 0$ ✓
(c) $f''(c) = 0$ (d) $f'(c) \neq 0$

8- $\int \sec^2 x \, dx =$:

(a) $\cot x + c$ (b) $\tan x + c$ ✓
(c) $2 \sec x + c$ (d) $\frac{1}{\cos^2 x} + c$

9- $\int_{-\pi}^{\pi} \sin x \, dx =$:

(a) 2π (b) 0 ✓
(c) 1 (d) $\cos \pi$

10- If $f(x) = 2x + 1$, then $f^{-1}(x) = ?$:

(a) $2x - 1$ (b) $1 - 2x$
(c) $x - \frac{1}{2}$ (d) $\frac{x-1}{2}$ ✓

11- y-intercept of the line $2x - y - 4 = 0$ is:

(a) 2 (b) −2
(c) 4 (d) −4 ✓

12- An angle in the semi-circle is of measure:

(a) 30° (b) 60°
(c) 90° ✓ (d) 180°

13- The perpendicular distance of a line $5x + 12y = 7$ from origin is:

(a) $\frac{1}{13}$ (b) $\frac{13}{7}$
(c) $\frac{7}{13}$ ✓ (d) −7

14- Equation of latus-rectum of parabola $y^2 = 4ax$ is:

(a) $x = -a$ (b) $y = -a$
(c) $x = a$ ✓ (d) $y = a$

15- The mid-point of line segment joining A(−8, 3), B(2, −1) is:

(a) (−6, 2) (b) (10, 4)
(c) (−3, 1) ✓ (d) (−16, −3)

16- **The triple scalar product of vectors, calculates the volume of:**

(a) Triangle (b) Parallelogram
(c) Tetrahedron (d) Parallelepiped √

17- **The equation of line $\frac{x}{b} + \frac{y}{a} = 1$ is in:**

(a) Normal form (b) Intercept form √
(c) Point-slope form (d) Two-points form

18- **The radius of circle $x^2 + y^2 = 5$ is:**

(a) 25 (b) $\sqrt{5}$ √
(c) 5 (d) (0, 0)

19- **Non-zero vector $\underline{a}$ and $\underline{b}$ are parallel, if $\underline{a} \times \underline{b} =$:**

(a) 0 √ (b) 1
(c) −1 (d) (a, b)

20- **The solution of the inequality $x + 2y < 6$ is:**

(a) (1, 1) √ (b) (1, 3)
(c) (1, 4) (d) (1, 5)

Inter (Part-II) 2019

Mathematics	Group-II	PAPER: II
Time: 2.30 Hours	(SUBJECTIVE TYPE)	Marks: 80

SECTION-I

2. Write short answers to any EIGHT (8) questions: 16

(i) Define implicit function.

Ans If x and y are so mixed up and y cannot expressed in terms of the independent variable x, then y is called an implicit function. For example,

1. $x^2 + xy + y^2 = 0$
2. $\frac{xy^2 - y + 9}{xy} = +1$ are implicit functions of x and y.

Symbolically, it is written as $f(x, y) = 0$.

(ii) $f(x) = 3x^4 - 2x^2$, $g(x) = \frac{2}{\sqrt{x}}$, find $f(g(x))$.

Ans $f(x) = 3x^4 - 2x^2$ $\qquad g(x) = \frac{2}{\sqrt{x}}$

$$fog(x) = f[g(x)] = 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left(\frac{4}{\sqrt{x}}\right)^2$$

$$= 3\left(\frac{16}{x^2}\right) - 2\left(\frac{4}{x}\right)$$

$$= \frac{48}{x^2} - \frac{8}{x}$$

$$= \frac{48 - 8x}{x^2}$$

$$= \frac{8(6 - x)}{x^2}$$

(iii) Evaluate $\lim_{x \to 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2}$.

Ans

$$= \lim_{x \to 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} \times \frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} + \sqrt{2}}$$

$$= \lim_{x \to 2} \frac{x - 2}{(x - 2)(\sqrt{x} + \sqrt{2})}$$

$$= \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

(iv) **Find derivative by definition of x^2.**

Ans $f(x) = x^2$

1. $f(x + \delta x) = (x + \delta x)^2$

2. $f(x + \delta x) - f(x) = (x + \delta x)^2 - x^2$

$$= x^2 + 2x\,\delta x + (\delta x)^2 - x^2$$

$$= 2x\,\delta x + (\delta x)^2 = (2x + \delta x)\,\delta x$$

3. $\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{(2x+\delta x)\,\delta x}{\delta x} = 2x + \delta x \qquad (\delta x \neq 0)$

4. $\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$

$\lim_{\delta x \to 0} (2x + \delta x) = 2x$

i.e., $f'(x) = 2x$

(v) **Differentiate w.r.t. 'x' $\sqrt{\frac{a-x}{a+x}}$.**

Ans Let $y = \sqrt{\frac{a-x}{a+x}}$ and $u = \frac{a-x}{a+x}$

Then $y = u^{1/2}$

Now, $\frac{dy}{du} = \frac{1}{2}u^{1/2-1} = \frac{1}{2}u^{-1/2}$

and $\frac{du}{dx} = \frac{d}{dx}\left[\frac{a-x}{a+x}\right]$

$$= \frac{\left[\frac{d}{dx}(a-x)\right](a+x) - (a-x)\left[\frac{d}{dx}(a+x)\right]}{(a+x)^2}$$

$$= \frac{(0-1)(a+x) - (a-x)(0+1)}{(a+x)^2} = \frac{-a-x-a+x}{(a+x)^2}$$

$$= \frac{-2a}{(a+x)^2}$$

Using the formula,

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, we have

$$\frac{d}{dx}\left(\sqrt{\frac{a-x}{a+x}}\right)=\frac{1}{2}u^{-1/2}\left[\frac{-2a}{(a+x)^2}\right]$$

$$=\frac{1}{2}\left(\frac{a-x}{a+x}\right)^{1/2}\times\frac{-2a}{(a+x)^2} \qquad \left(\therefore\ u=\frac{a-x}{a+x}\right)$$

$$=\frac{(a-x)^{-1/2}}{(a+x)^{-1/2}}\times\frac{-a}{(a+x)^2}=\frac{-a}{(a-x)^{1/2}(a+x)^{3/2}}$$

(vi) Find $\frac{dy}{dx}$, if $x^2-4xy-5y=0$.

Ans $x^2-4xy-5y=0$

Differentiating the equation w.r.t x:

$$\frac{d}{dx}(x^2-4xy-5y)=0$$

$$\frac{d}{dx}x^2-\frac{d}{dx}(4xy)-\frac{d}{dx}(5y)=0$$

$$2x-4\left[x\frac{d}{dx}(y)+y\frac{d}{dx}(x)\right]-5\frac{d}{dx}(y)=0$$

$$2x-4\left(x\frac{dy}{dx}+y\right)-5\frac{dy}{dx}=0$$

$$2x-4x\frac{dy}{dx}-4y-5\frac{dy}{dx}=0$$

$$-\frac{dy}{dx}(4x+5)=4y-2x$$

$$\frac{dy}{dx}=-\frac{(4y-2x)}{4x+5}$$

$$=\frac{2x-4y}{4x+5}=\frac{2(x-2y)}{4x+5}$$

(vii) Prove that $\frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$.

Ans Let $y=\cot^{-1}x \Rightarrow y=\frac{1}{\cot}x$

$\therefore \quad \cot y = x$ (i)

Differentiating w.r.t. x,

$$\frac{d}{dx}(\cot y)=\frac{d}{dx}(x)$$

$$-\text{cosec}^2 y\cdot\frac{dy}{dx}=1$$

$$\therefore \quad \frac{dy}{dx}=-\frac{1}{\text{cosec}^2 y}$$

$$= \frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2} \quad \text{Proved.}$$

(viii) **Find $\frac{dy}{dx}$, if $y = x \cos y$.**

Ans $y = x \cos y$

Differentiate w.r.t x.

$$\frac{dy}{dx} = \frac{d}{dx}(x \cos y)$$

$$= x\frac{d}{dx}(\cos y) + \cos y \frac{d}{dx}(x)$$

$$= x\left(-\sin y \cdot \frac{dy}{dx}\right) + \cos y\,(1)$$

$$= -x \sin y \frac{dy}{dx} + \cos y$$

$$= \frac{dy}{dx} + x \sin y \frac{dy}{dx} = \cos y$$

$$= \frac{dy}{dx}(1 + x \sin y) = \cos y$$

$$\frac{dy}{dx} = \frac{\cos y}{1 + x \sin y}$$

(ix) **Find $f'(x)$, if $f(x) = \sqrt{\ln(e^{2x} + e^{-2x})}$.**

Ans Let $u = e^{2x} + e^{-2x}$

Then $f(x) = (\ln u)^{1/2}$ and

$$f'(x) = \frac{d}{dx}(\ln u)^{1/2} = \frac{d}{du}(\ln u)^{1/2} \times \frac{du}{dx} = \left[\frac{1}{2}(\ln u)^{1/2-1}\frac{d}{du}(\ln u)\right] \times \frac{d}{dx}(e^{2x} + e^{-2x})$$

$$= \left(\frac{1}{2} \cdot \frac{1}{(\ln u)^{1/2}} \cdot \frac{1}{u}\right) \cdot (e^{2x} \cdot 2 + e^{-2x} \cdot (2))$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{\ln(e^{2x} + e^{-2x})}} \cdot \frac{1}{e^{2x} + e^{-2x}} \cdot 2(e^{2x} - e^{-2x})$$

$$= \frac{e^{2x} + e^{-2x}}{\sqrt{\ln(e^{2x} + e^{-2x})} \times (e^{2x} + e^{-2x})}$$

(x) **Find y_2, if $x = at^2$, $y = bt^4$.**

Ans $x = at^2$; $y = bt^4$

$\frac{dx}{dt} = \frac{d}{dt}(at^2)$; $\frac{dy}{dt} = \frac{d}{dt}(bt^4)$

$= 2at$; $= 4bt^3$

3. **Write short answers to any EIGHT (8) questions: 16**

(i) **Using differential, find $\frac{dy}{dx}$ when $xy - \ln x = c$.**

Ans

$$xy - \ln x = c$$

Taking differential on both sides,

$$d(xy - \ln x) = d(c)$$

$$x\,dy + y\,dx - \frac{1}{x}dx = 0$$

$$x\,dy + y\,dx - \frac{1}{x}dx = 0$$

$$x\,dy = \frac{dx}{x} - y\,dx$$

$$= dx\left(\frac{1}{x} - y\right)$$

$$\frac{dy}{dx} = \frac{\frac{1-xy}{x}}{x} = \frac{1-xy}{x^2}$$

(ii) **Evaluate $\int \frac{(\sin x + \cos^2 x)}{\cos^2 x \cdot \sin x}dx$.**

Ans

$$= \int\left(\frac{\sin x}{\cos^2 x \cdot \sin x} + \frac{\cos^2 x}{\cos^2 x \cdot \sin x}\right)dx$$

$$= \int\left(\frac{1}{\cos^2 x} + \frac{1}{\sin x}\right)dx$$

$$= \int \sec^2 x\,dx + \int \operatorname{cosec} x\,dx$$

$$= \tan x + \ln|\operatorname{cosec} x - \cot x| + c$$

(iii) **Find $\int x(\sqrt{x} + 1)\,dx$; $x > 0$.**

Ans

$$\int x \cdot x^{1/2}\,dx + \int x\,dx$$

$$\int x^{3/2}\,dx + \frac{x^2}{2} + c$$

$$= \frac{x^{5/2}}{\frac{5}{2}} + \frac{x^2}{2} + c$$

$$= \frac{2x^{5/2}}{5} + \frac{x^2}{2} + c$$

By Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 4bt^3 \times \frac{1}{2at}$$

$$\therefore \quad y_1 = \frac{2bt^2}{a}$$

$$\frac{d^2y}{dx^2} = \frac{2b}{a} \cdot \frac{d}{dx}(t^2)$$

$$\therefore \quad y_2 = \frac{2b}{a} \cdot (2t) \cdot \frac{d}{dx}(t)$$

$$= \frac{4bt}{a} \cdot \frac{dt}{dx}$$

$$= \frac{4bt}{a} \times \frac{1}{2at} = \frac{2b}{a^2}$$

(xi) Define Maclaurin series.

Ans We have $a_0 = f(0)$, $a_1 = f'(0)$, $a_2 = f'' \frac{(0)}{2!}$

$$a_3 = \frac{f'''(0)}{3!}, a_4 = \frac{f''''(0)}{4!}$$

Following the above pattern, we can write

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Thus substituting these values in the power series, we have

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^n(0)}{n!}x^n + \dots$$

This expansion of f(x) is called the Maclaurin series.

(xii) Determine the interval in which f(x) is increasing or decreasing if f(x) = sin x, x ∈ (0, π).

Ans

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

$f'(x)$ is +ve $\forall\, x \in (0, \frac{\pi}{2})$

$\therefore$ f(x) is increasing.

$f'(x) = \cos x$ is –ve for all $x \in (\frac{\pi}{2}, \pi)$

then t is decreasing.

(iv) **Evaluate $\int a^{x^2} . x\, dx$; $a > 1$.**

Ans Put $x^2 = t$, then $x\, dx = \frac{1}{2} dt$

Thus $\int a^{x^2} . x\, dx = \int a^t \times \frac{1}{2} dt$

$$= \frac{1}{2} \int a^t dt = \frac{1}{2} \frac{a^t}{\ln a} + c$$

$$= \frac{a^{x^2}}{2 \ln a} + c$$

(v) **Find the anti-derivative of $x . e^x$.**

Ans $\int x . e^x . dx$ Let $u = x$ and $dv = e^x dx$

Then $du = 1 . dx$ and $v = e^x$

Applying the formula for integration by parts, we have

$$\int x e^x dx = x e^x - \int e^x \times 1\, dx$$

$$= x e^x - e^x + c$$

(vi) **Evaluate $\int e^x (\cos x + \sin x)\, dx$.**

Ans $= \int e^x \cos x\, dx + \int e^x \sin x\, dx$

Integrating 1st integral by parts,

$$= e^x \int \cos x\, dx - \int \left[\frac{d}{dx}(e^x) . \int \cos x\, dx\right] dx + \int e^x \sin x\, dx + c$$

$$= e^x . \sin x - \int e^x . \sin x\, dx + \int e^x \sin x\, dx + c$$

$$= e^x \sin x + c$$

(vii) **State 'Fundamental Theorem' of calculus.**

Ans If f is continuous on [a , b] and $\phi'(x) = f(x)$, that is, $\phi(x)$ is any anti-derivative of f on [a, b], then

$$\int_a^b f(x)\, dx = \phi(b) - \phi(a)$$

Note that the difference $\phi(b) - \phi(a)$ is independent of the choice of anti-derivative of the function f.

(viii) **Compute $\int_{-1}^{1} (x^{1/3} + 1)\, dx$.**

Ans $\int_{-1}^{1} (x^{1/3} + 1)\, dx = \int_{-1}^{1} x^{1/3} dx + \int_{-1}^{1} dx$

$$= \left[\frac{x^{4/3}}{\frac{4}{3}}\right]_{-1}^{1} + [x]_{-1}^{1}$$

$$= \frac{3}{4}\left[(1)^{4/3} - (-1)^{4/3}\right] + [(1) - (-1)]$$

$$= \frac{3}{4}[1 - 1] + 2 = 2$$

(ix) Find the area above x-axis and under the curve $y = 5 - x^2$ from $x = -1$ to $x = 2$.

Ans Area $= \int_{-1}^{2} (5 - x^2)\, dx$

$$= 5\int_{-1}^{2} dx - \int_{-1}^{2} x^2\, dx$$

$$= 5[x]_{-1}^{2} - \left[\frac{x^3}{3}\right]_{-1}^{2}$$

$$= 5[2 - (-1)] - \frac{1}{3}[8 - (-1)^3]$$

$$= 5[3] - \frac{1}{3}[8 + 1]$$

$$= 15 - \frac{1}{3}(9) = 15 - 3 = 12 \text{ sq. units}$$

(x) Solve the differential equation $\sin y . \operatorname{cosec} x . \frac{dy}{dx} = 1$.

Ans

$$\sin y \operatorname{cosec} x \frac{dy}{dx} = 1$$

$$\sin y\, dy = \frac{dx}{\operatorname{cosec} x}$$

$$\sin y\, dy = \sin x\, dx$$

$$\int \sin y\, dy = \int \sin x\, dx + c'$$

$$-\cos y = -\cos x + c'$$

$$\cos y = \cos x - c'$$

$$\cos y = \cos x + c$$

(xi) Define 'decision variables'.

Ans The variables used in the system of linear inequalities relating to the problems of everyday life are non-negative constraints. These non-negative constraints play an important role for taking decision. So these variables are decision variables.

(xii) Graph solution set of inequality $2x + y \geq 2$ in $x - y$ plane.

Ans We draw the graph of the line $2x + y = 2$ joining the points (1, 0) and (0, 2). The point (0, 0) does not satisfy the inequality $2x + y > 0$ because $2(0) + 0 = 0 \not> 2$. Thus the graph of the inequality $2x + y \geq 2$ is the closed half not on the origin-side of the line $2x + y = 2$.

The intersection point (2, −2) can be found by solving the equation $2x + y = 2$.

4. Write short answers to any NINE (9) questions: 18

(i) Find the coordinates of the point that divides the join of A(−6, 3) and B(5, −2) internally in ratio 2 : 3.

Ans Here $k_1 = 2, k_2 = 3, x_1 = -6, x_2 = 5$

By the formula, we have $y_1 = 3, y_2 = -2$

$$x = \frac{(2 \times 5) + 3(-6)}{2+3} = \frac{10-18}{5} = \frac{-8}{5}$$

and $$y = \frac{(2(-2)) + 3 \times 3}{2+3} = \frac{-4+9}{5} = \frac{5}{5} = 1$$

Coordinates of the required points are $\left(\frac{-8}{5}, 1\right)$.

(ii) Find the slope and inclination of the line joining the points A(−2, 4) and B(5, 11).

Ans Let A(−2, 4) and B(5, 11)

$$\text{Slope of AB} = m = \frac{11-4}{5-(-2)} = \frac{7}{5+2} = \frac{7}{7} = 1$$

$$m = \tan\theta = 1 \Rightarrow \text{inclination} = \theta = \tan^{-1}(1) = 45^\circ$$

(iii) By means of slopes show that points A(−1, −3), B(1, 5) and C(2, 9) are collinear.

Ans We know that the points A, B and C are collinear, if the line AB and BC have the same slopes.

$$\text{Slope of AB} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-3)}{1-(-1)} = \frac{5+3}{1+2} = \frac{8}{2} = 4$$

and $$\text{Slope of BC} = \frac{9-5}{2-1} = \frac{4}{1} = 4$$

$\therefore$ Slope of AB = Slope of BC

Thus A, B and C are collinear.

(iv) **Find equation of the line through (–4, 7) and parallel to the line $2x - 7y + 4 = 0$.**

Ans
$$2x - 7y + 4 = 0$$
$$-7y = -2x - 4$$
$$7y = 2x + 4$$
$$\frac{7y}{7} = \frac{2x}{7} + \frac{4}{7}$$
$$y = \frac{2}{7}x + \frac{4}{7}$$

∴ Slope of required linear = $m = \frac{2}{7}$

Equation of line through (–4, 7) with $m = \frac{2}{7}$ is
$$y - y_1 = m(x - x_1)$$
$$y - 7 = \frac{2}{7}(x - (-4))$$
$$y - 7 = \frac{2}{7}(x + 4)$$
$$7y - 49 = 2x + 8$$
$$2x - 7y = 49 + 8 \Rightarrow 2x - 7y = 57$$
$$\Rightarrow 2x - 7y - 57 = 0$$

(v) **Find equation of circle with centre at (5, –2) and radius 4.**

Ans Here $c(h, k) = c(5, -2)$ and $r = 4$

Equation of required circle is
$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x - 5)^2 + (y - (-2)^2) = (4)^2$$
$$(x - 5)^2 + (y + 2)^2 = (4)^2$$
$$x^2 + 25 - 10x + y^2 + 4y + 4 = 16$$
$$x^2 + y^2 - 10x + 4y + 29 = 16$$
$$x^2 + y^2 - 10x + 4y + 29 - 16 = 0$$
$$x^2 + y^2 - 10x + 4y + 13 = 0$$

(vi) **Find focus and vertex of the parabola $y^2 = -8(x - 3)$.**

Ans $y^2 = -8(x - 3) \Rightarrow y^2 = -8x$ where $x = x - 3$

Comparing it with $y^2 = -4ax$
$$-4a = -8$$
$$4a = 8$$

$$a = \frac{8}{4} = 2$$

Coordinates of focus: $(-a, 0)$

$$x = -a, \ y = 0$$
$$x - 3 = -2$$
$$x = 3 - 2 = 1 \qquad f(1, 0)$$

Coordinates of vertex, $(0, 0)$

$$x = 0, \ y = 0$$
$$x - 3 = 0 \qquad v(0, 0)$$

(vii) Find equation of tangent to the parabola $x^2 = 16y$ at the point whose abscissa is 8.

Ans Since $x = 8$ lies on the parabola,

Substituting this value of x into the given equation, we find

$$x^2 = 16y$$
$$(8)^2 = 16y$$
$$64 = 16y \qquad \Rightarrow \qquad y = 4$$

Thus we have to find equations of tangent and normal at $(8, 4)$.

Slope of the tangent to the parabola at $(8, 4)$ is 1. An equation of the tangent the parabola at $(8, 4)$ is

$$y - 4 = x - 8$$
$$x - y + 4 - 8 = 0$$
$$x - y - 4 = 0$$

Slope of the normal at $(8, 4)$ is -1. Therefore, equation of the normal at the given point is

$$y - 4 = -(x - 8)$$
$$y - 4 = +8 - x$$
$$x + y - 4 - 8 = 0$$
$$x + y - 12 = 0$$

(viii) Find foci and vertices of the ellipse $25x^2 + 9y^2 = 225$.

Ans

$$25x^2 + 9y^2 = 225$$

Dividing both sides by 225

$$\frac{25x^2}{225} + \frac{9y^2}{225} = \frac{225}{225}$$
$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

Here

$$a^2 = 25 \qquad \Rightarrow \qquad a = \pm 5$$
$$b^2 = 9 \qquad \Rightarrow \qquad b = \pm 3$$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{25 - 9}{25} \Rightarrow \frac{16}{25} \Rightarrow e = \frac{4}{5}$$

$$ae = 5\left(\frac{4}{5}\right) = 4$$

$$\frac{a}{e} = \frac{5}{\frac{4}{5}} = \frac{5 \times 5}{4} = \frac{25}{4}$$

$$f = (0, \pm ae) = (0, \pm 4)$$

$$V(0, \pm a) = V(0 \pm 5)$$

(ix) Find the angle between the vectors $\underline{u} = 2\underline{i} - \underline{j} + \underline{k}$ and $\underline{v} = -\underline{i} + \underline{j}$.

Ans

$$\underline{u} = 2\underline{i} - \underline{j} + \underline{k}$$

$$\underline{v} = -\underline{i} + \underline{j}$$

$$\underline{u}.\underline{v} = (2\underline{i} - \underline{j} + \underline{k}) . (-\underline{i} + \underline{j} + 0\underline{k})$$

$$= (2)(-1) + (-1)(1) + (1)(0) = -3$$

$\therefore$ $$|\underline{u}| = |2\underline{i} - \underline{j} + \underline{k}| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{6}$$

and $$|\underline{v}| = |-\underline{i} + \underline{j} + 0k| = \sqrt{(-1)^2 + (1)^2 + (0)^2} = \sqrt{2}$$

$$\cos\theta = \frac{\underline{u} . \underline{v}}{|\underline{u}| . |\underline{v}|}$$

$$\cos\theta = \frac{-3}{\sqrt{6}\sqrt{2}} - \frac{\sqrt{3}}{2}$$

$\therefore$ $$\theta = \frac{5\pi}{6}$$

(x) Find scalar α so that the vectors $2\underline{i} + \alpha\,\underline{j} + 5\underline{k}$ and $3\underline{i} + \underline{j} + \alpha\,\underline{k}$ are perpendicular.

Ans Let $\underline{u} = 2\underline{i} + \alpha\underline{j} + 5\underline{k}$

and $\underline{v} = 3\underline{i} + \underline{j} + \alpha\underline{k}$

It is given that $\underline{u}$ and $\underline{v}$ are perpendicular.

$\therefore$ $\underline{u} . \underline{v} = 0$

$\Rightarrow$ $(2\underline{i} + \alpha\underline{j} + 5\underline{k}) . (3\underline{i} + \underline{j} + \alpha\,\underline{k}) = 0$

$\Rightarrow$ $6 + \alpha + 5\alpha = 0$

$$6\alpha = -6$$

$$\alpha = \frac{-6}{6} = -1$$

$\therefore$ $$\alpha = -1$$

(xi) **If $\underline{v}$ is a vector for which $\underline{v} \cdot \underline{i} = 0$, $\underline{v} \cdot \underline{j} = 0$, $\underline{v} \cdot \underline{k} = 0$, find $\underline{v}$.**

Ans Let $\underline{v} = a\underline{i} + bj + c\underline{k}$ (i)

$\because \underline{v} \cdot \underline{i} = 0$

$\therefore (a\underline{i} + b\underline{j} + c\underline{k})(\underline{i} - 0\underline{j} + 0\underline{k}) = 0$

$(a)(1) + (b)(0) + (c)(0) = 0$

$a = 0$ (ii)

$\because \underline{v} \cdot \underline{j} = 0$

$\therefore (a\underline{i} + b\underline{j} + c\underline{k}) \cdot (0\underline{i} + \underline{j} + 0\underline{k}) = 0$

$\therefore (a)(0) + b(1) + c(0) = 0$

$\therefore b = 0$ (iii)

$\because \underline{v} \cdot \underline{k} = 0$

$\therefore (a\underline{i} + b\underline{j} + c\underline{k}) \cdot (0\underline{i} + 0\underline{j} + \underline{k}) = 0$

$\therefore (a \times 0) + (b \times 0) + (c \times 1) = 0$

$\therefore c = 0$ (iv)

Putting (ii) (iii) and (iv) in eq. (i),

$v = (0)\underline{i} + 0(\underline{j}) + 0(\underline{k})$

$\underline{v} = 0$ (Zero / Null vector)

(xii) **Prove that $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = 0$.**

Ans $\underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} + \underline{b} \times \underline{a} + \underline{c} \times \underline{a} + \underline{c} \times \underline{b} = 0$

$\underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} - \underline{a} \times \underline{b} - \underline{a} \times \underline{c} - \underline{b} \times \underline{c} = 0$

$= 0$ R.H.S

(xiii) **Find the value of α, so that $\alpha\underline{i} + \underline{j}$, $\underline{i} + \underline{j} + 3\underline{k}$ and $2\underline{i} + \underline{j} - 2\underline{k}$ are coplanar.**

Ans Let $\underline{u} = a\underline{i} + \underline{j}$ $\quad \underline{v} = \underline{i} + \underline{j} + 3\underline{k}$ and

$\underline{w} = 2\underline{i} + \underline{j} - 2\underline{k}$

Triple scalar product

$$[\underline{u}\ \underline{v}\ \underline{w}] = \begin{bmatrix} \alpha & 1 & 0 \\ 1 & 1 & 3 \\ 2 & 1 & -2 \end{bmatrix}$$

$= \alpha(-2 - 3) + -1(-2 - 6) + 0(1 - 2)$

$= -5\alpha + 8$

The vectors will be coplaner if

$-5\alpha + 8 = 0$

$-5\alpha = -8$

$$\alpha = \frac{8}{5}$$

SECTION-II

NOTE: Attempt any Three (3) questions.

Q.5.(a) If $f(x) = \begin{cases} 3x & \text{if } x \le -2 \\ x^2 - 1 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \ge 2 \end{cases}$ **(5)**

discuss continuity at x = 2 and x = –2.

Ans (i) At $x = 2$

(a) $f(2) = 3$ (Defined)

(b) L.H.L. $= \lim_{x \to 2^-} (x^2 - 1) = (2)^2 - 1 = 3$

R.H.L. $= \lim_{x \to 2^+} (3) = 3$

$\therefore$ L.H.L. = R.H.L.

(c) $\lim_{x \to 2} f(x) = 3$

$\therefore$ $\lim_{x \to 2} f(x) = f(2)$

Hence f(x) is continuous of x = 2.

(ii) At $x = -2$

(a) $f(-2) = 3(-2) = -6$ (Defined)

(b) L.H.L. $= \lim_{x \to -2^-} 3x = 3(-2) = -6$

R.H.L $= \lim_{x \to -2^+} (x^2 - 1) = (-2)^2 - 1 = 3$

$\therefore$ L.H.L $\neq$ R.H.L.

Hence f(x) is discontinuous at x = –2. There is no need to investigate (c).

(b) If $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$. (5)

Ans

$$y = e^x \sin x \quad \text{(i)}$$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \sin x)$$

$$= e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x)$$

$$= e^x \cos x + e^x \sin x \quad \text{(ii)}$$

$$= e^x (\sin x + \cos x)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[e^x(\sin x + \cos x)]$$
$$= e^x \frac{d}{dx}(\sin x + \cos x) + (\sin x + \cos x)\frac{d}{dx}e^x$$
$$= e^x(\cos x - \sin x) + e^x(\sin x + \cos x)$$
$$= e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x$$
$$= 2e^x \cos x \qquad \text{(iii)}$$

From (ii),

$$\frac{dy}{dx} = e^x \cos x + e^x \sin x$$
$$= e^x \cos x + y \qquad \text{(From (i))}$$
$$e^x \cos x = \frac{dy}{dx} - y$$

Putting (iv) in (iii),

$$\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx} - y\right)$$
$$= 2\frac{dy}{dx} - 2y$$
$$\frac{d^2y}{dx^2} - 2\frac{2dy}{dx} + 2y = 0 \qquad \text{Proved.}$$

Q.6.(a) Integrate $\int \frac{12}{x^3 + 8} dx$. (5)

Ans $\frac{12}{x^3 + 8} = \frac{12}{(x)^3 + (2)^3} = \frac{12}{(x + 2)(x^2 - 2x + 4)}$

Making partial fractions,

$$\frac{12}{(x+2)(x^2 - 2x + 4)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 - 2x + 4} \qquad \text{(i)}$$
$$12 = A(x^2 - 2x + 4) + (Bx + C)(x + 2) \qquad \text{(ii)}$$

For A, let $x + 2 = 0$

$$x = -2$$

Putting $x = -2$ in eq (ii),

$$12 = A((-2)^2 - 2(-2) + 4) + 0$$
$$12 = A(4 + 4 + 4) + 0$$
$$12 = 12A$$
$$\frac{12}{12} = A$$
$$\boxed{1 = A}$$

Expanding eq. (ii),

$$12 = Ax^2 - 2Ax + 4A + Bx^2 + 2Bx + Cx + 2C$$
$$= (A + B)x^2 + (-2A + 2B + C)x + (4A + 2C)$$

Comparing coefficients on both sides,

$A + B = 0$	$-2A + 2B + C = 0$	$4A + 2C = 12$
$1 + B = 0$	$-2(1) + 2(-1) + C = 0$	
$\boxed{B = -1}$	$\boxed{C = 4}$	

Putting the values of A, B and C in (i),

$$\frac{12}{(x+2)(x^2-2x+4)} = \frac{1}{x+2} + \frac{-x+4}{x^2-2x+4}$$
$$= \frac{1}{x+2} - \frac{x-4}{x^2-2x+4}$$

$$\int \frac{12}{(x+2)(x^2-2x+4)} = \int \frac{d}{x+2} - \int \frac{x-1-3}{x^2-2x+4}\,dx$$
$$= \ln|x+2| - \frac{1}{2}\int \frac{2x-2}{x^2-2x+4}\,dx + 3\int \frac{dx}{x^2-2x+4}$$
$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + 3\int \frac{dx}{x^2-2.x.1+(1)^2+3}$$
$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + 3\int \frac{dx}{(x-1)^2+(\sqrt{3})^2}$$
$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + 3\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)\right] + C$$
$$= \ln|x+2| - \frac{1}{2}|x^2-2x+4| + \sqrt{3}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + C$$

(b) Find equations of two parallel lines, perpendicular to $2x - y + 3 = 0$ such that the product of the x- and y-intercepts of each is 3. **(5)**

Ans $2x - y + 3 = 0$
$$-y = -2x - 3$$
$$y = 2x + 3$$

$\therefore$ Slope of required lines $= m = -\frac{1}{2}$

Equation of required lines are:

$$y = mx + c$$
$$= -\frac{1}{2}x + c = \frac{-x+2c}{2}$$
$$2y = -x + 2c$$

or $\quad x + 2y - 2c = 0 \quad$ (i)

For x-intercept, put y = 0 in (i),

$\therefore \quad x + 0 - 2c = 0$

$x = 2c$

For y-intercept, put x = 0 in (i),

$2y - 2c = 0$

$2y = 2c \Rightarrow y = c$

As product of x and y-intercept = 3

$(2c)(c) = 3$

$2c^2 = 3$

$c^2 = \frac{3}{2}$

$c = \pm\sqrt{\frac{3}{2}}$

Putting it in (1),

$x + 2y - 2\left(\pm\sqrt{\frac{3}{2}}\right) = 0$

$\Rightarrow \quad x + 2y \pm \sqrt{6} = 0$

$\therefore \quad x + 2y + \sqrt{6} = 0 \quad$ and $\quad x + 2y - \sqrt{6} = 0$

Q.7.(a) Evaluate the definite integral $\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x(2 + \sin x)} dx$. (5)

Ans Let $\sin x = t \Rightarrow \cos x\, dx = dt$

Limits: Let $x = \frac{\pi}{6} \Rightarrow t = \sin\frac{\pi}{6} = \frac{1}{2}$

Let $x = \frac{\pi}{2} \Rightarrow t = \sin\frac{\pi}{2} = 1$

$$\int_{\pi/6}^{\pi/2} \frac{\cos x\, dx}{\sin x(2 + \sin x)} = \int_{1/2}^{1} \frac{dt}{t(2 + t)} \quad \text{(i)}$$

Making partial fractions,

$$\frac{1}{t(2 + t)} = \frac{A}{t} + \frac{B}{2 + t} \quad \text{(ii)}$$

$$1 = A(2 + t) + B(t) \quad \text{(iii)}$$

For A, let t = 0

$1 = A(2 + 0)$

$2A = 1$

$\boxed{A = \frac{1}{2}}$

For B, let $2 + t = 0$

$$t = -2$$

Putting the value of t in eq. (iii),

$$1 = 0 - 2B \Rightarrow B = -\frac{1}{2}$$

Putting the value of A and B in eq. (ii),

$$\frac{1}{t(2+t)} = \frac{1}{2t} - \frac{1}{2(2+t)}$$

Putting it in eq. (i),

$$\int_{\pi/6}^{\pi/2} \frac{\cos x\, dx}{\sin x(2+\sin x)} = \int_{1/2}^{1} \frac{dt}{t(2+t)}$$

$$= \frac{1}{2}\int_{1/2}^{1} \frac{dt}{t} - \frac{1}{2}\int_{1/2}^{1} \frac{dt}{2+t}$$

$$= \frac{1}{2}[\ln|t|]_{1/2}^{1} - \frac{1}{2}[\ln|2+t|]_{1/2}^{1}$$

$$= \frac{1}{2}\left[\ln(1) - \ln\left(\frac{1}{2}\right)\right] - \frac{1}{2}\left[\ln(2+1) - \ln\left(2+\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}[0 - \ln(2)^{-1}] - \frac{1}{2}\left[\ln(3) - \ln\left(\frac{5}{2}\right)\right]$$

$$= \frac{1}{2}[\ln(2)] - \frac{1}{2}[\ln(3) - \ln(5) + \ln(2)]$$

$$= \frac{1}{2}[\ln(2) - \ln(3) + \ln(5) - \ln(2)]$$

$$= \frac{1}{2}\ln\left(\frac{5}{3}\right)$$

(b) Minimize $z = 2x + y$ subject to the constraints **(5)**

$$x + y \geq 3, \quad 7x + 5y \leq 35, \quad x \geq 0, \quad y \geq 0$$

Ans

$x + y \geq 3$ (i)

$x + y = 3$ (iii)

Putting $x = 0$ in (iii),

$0 + y = 3 \Rightarrow y = 3$

$\therefore$ (0, 3) is a point on (iii).

Putting $y = 0$ in (iii),

$x + 0 = 3 \Rightarrow x = 3$

$\therefore$ (3, 0) is another point on (iii).

Putting $x = 0, y = 0$ in (i),

$7x + 5y \leq 35$ (ii)

$7x + 5y = 35$ (iv)

Putting $y = 0$ in (iv),

$0 + 5y = 35 \Rightarrow y = 7$

$\therefore$ (0, 7) is a point on (iv).

Putting $x = 0$ in (iv),

$7x + 0 = 35 \Rightarrow x = 5$

$\therefore$ (5, 0) is another point on (iv).

Putting $x = 0, y = 0$ in (ii),

$$0 + 0 > 3$$
$$0 > 3$$

Which is false. Hence solution region of (i) does not lie on the origin-side of (i).

$$0 + 0 < 35$$
$$\therefore \quad 0 < 35$$

Which is true. Hence solution region of (ii) lies on the origin-side of (ii).

Q.8.(a) Find equation of the line through the point (2, –9) and intersection of the lines **(5)**

$$2x + 5y - 8 = 0$$
$$3x - 4y - 6 = 0$$

Ans

$$2x + 5y - 8 = 0 \quad \text{(i)}$$

From (i),

$$2x = 8 - 5y$$
$$x = \frac{8 - 5y}{2} \quad \text{(iii)}$$

$$3x - 4y - 6 = 0 \quad \text{(ii)}$$

Putting eq. (iii) to eq. (ii),

$$3\left(\frac{8 - 5y}{2}\right) - 4y = 6$$
$$\frac{24 - 15y - 8y}{2} = 6$$
$$-23y + 24 = 12$$
$$-23y = -12$$
$$\therefore \quad y = \frac{12}{23}$$

Putting the value of y in eq. (iii),

$$x = \frac{8 - 5\left(\frac{12}{23}\right)}{2}$$
$$= \frac{\frac{184 - 60}{23}}{2}$$
$$= \frac{124}{46} = \frac{62}{23}$$

$\therefore \quad \left(\frac{62}{23}, \frac{12}{23}\right)$ is the point of intersection of (i) and (ii).

Equation of line through (2, –9) and $\left(\frac{62}{23}, \frac{12}{23}\right)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y+9=\frac{\frac{12}{23}+9}{\frac{62}{23}-2}(x-2)$$

$$=\frac{\frac{12+207}{23}}{\frac{62-46}{23}}(x-2)$$

$$y+9=\frac{219}{16}(x-2)$$

$$16y+144=219x-438$$

$$219x-16y-582=0$$

(b) Show that the circles $x^2+y^2+2x-2y-7=0$ and $x^2+y^2-6x+4y+9=0$ touch externally. (5)

Ans $x^2+y^2+2x-2y-7=0$

$$x^2+y^2+2(1)x+2(-1)y+(-7)=0$$

$$g_1=1,\ f_1=-1,\ c_1=-7$$

$$c_1'(-g_1,-f_1)=c_1'(-1,1)$$

$$r_1=\sqrt{g_1^2+f_1^2-c_1}$$

$$=\sqrt{(1)^2+(-1)^2-(-7)}$$

$$=\sqrt{1+1+7}=\sqrt{9}=3$$

$$x^2+y^2-6x+4y+9=0$$

$$x^2+y^2+2(-3)x+2(2)y+9=0$$

$$g_2=-3,\ f_2=(2),\ c_2=9$$

$$c_2'(-g_2,-f_2)=c_2'(3,-2)$$

$$r_2=\sqrt{g_2^2+f_2^2-c_2}$$

$$=\sqrt{(-3)^2+(2)^2-9}$$

$$=\sqrt{9+4-9}=\sqrt{4}=2$$

$$|c_1'c_2'|=r_1+r_2=3+2=5 \quad \text{(i)}$$

Also,

$$|c_1'c_2'|=\sqrt{(3+1)^2+(-2-1)^2}$$

$$=\sqrt{16+9}=\sqrt{25}=5 \quad \text{(ii)}$$

From (i) and (ii), it is prove that the given two circles touch externally.

Q.9.(a) Find an equation of the ellipse having foci $(\pm 5, 0)$ and passing through the point $\left(\frac{2}{3}, \sqrt{3}\right)$. (5)

Ans $F(\sqrt{5}, 0),\ F'(-\sqrt{5}, 0)$ through $\left(\frac{3}{2}, \sqrt{3}\right)$

$$a_e=\sqrt{5} \quad \Rightarrow \quad e=\frac{\sqrt{5}}{a}$$

$$e^2 = \frac{a^2 - b^2}{a^2} \Rightarrow \frac{5}{a^2} \frac{a^2 - b^2}{a^2}$$

$$= a^2 - b^2 = 5 \Rightarrow a^2 = 5 + b^2 \quad \text{(i)}$$

Equation of required ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{(ii)}$$

Since $\left(\frac{3}{2}, \sqrt{3}\right)$ lie on it, therefore

$$\frac{\frac{9}{4}}{a^2} + \frac{3}{b^2} = 1 \Rightarrow \frac{9}{4a^2} + \frac{3}{b^2} = 1 \Rightarrow \frac{9b^2 + 12a^2}{4a^2b^2} = 1$$

$$\therefore \quad 12a^2 + 9b^2 = 4a^2b^2 \quad \text{(iii)}$$

Putting the value of eq. (i) in eq. (iii),

$$12(5 + b^2) + 9b^2 = 4b^2(5 + b^2)$$

$$60 + 12b^2 + 9b^2 = 20b^2 + 4b^4$$

$$0 = 4b^4 - b^2 - 60$$

Let $b^2 = t$

$$4t^2 - t - 60 = 0$$

$$t = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-60)}}{2 \times 4} = \frac{1 \pm \sqrt{1 + 960}}{8}$$

$$= \frac{1 \pm 31}{8} \Rightarrow \frac{1 + 31}{8} \quad \& \quad t = \frac{1 - 31}{8}$$

$$= \frac{32}{8} \qquad = -\frac{30}{8}$$

$$t = 4, -\frac{15}{4} \qquad \therefore -\frac{15}{4}$$

Gives imaginary roots, hence discard it.

$$\therefore \quad b^2 = t = 4 \Rightarrow b = \pm 2$$

Putting the values in (i),

$$a^2 = 5 + 4 = 9 \Rightarrow a = \pm 3$$

Putting the values in (ii),

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

(b) A particle acted upon by constant forces $4\underline{i} + \underline{j} - 3\underline{k}$ and $3\underline{i} - \underline{j} - \underline{k}$ is displaced from A(1, 2, 3) to B(5, 4, 1). Find the work done. **(5)**

Ans Let $\vec{F_1} = 4\underline{i} + \underline{j} - 3\underline{k}$ and $\vec{F_2} = 3\underline{i} - \underline{j} - \underline{k}$

Total force $= \vec{F} = \vec{F_1} + \vec{F_2}$

$= (4\underline{i} + \underline{j} - 3\underline{k}) + (3\underline{i} - \underline{j} - \underline{k})$

$= 7\underline{i} + 0\underline{j} - 4\underline{k}$

$\vec{d} = \vec{AB} = (5 - 1)\,\underline{i} + (4 - 2)\,\underline{j} + (1 - 3)\,\underline{k}$

$= 4\underline{i} + 2\underline{j} - 2\underline{k}$

Work done $= \vec{F} . \vec{d} = (7\underline{i} + 0\underline{j} - 4\underline{k})(4\underline{i} + 2\underline{j} - 2\underline{k})$

$= (7 \times 4) + (0 \times 2) + (-4 \times -2)$

$= 28 + 0 + 8 = 36$