

## Inter (Part-II) 2017

Mathematics

Group-II

PAPER: II

Time: 30 Minutes

(OBJECTIVE TYPE)

Marks: 20

**Note:** Four possible answers, A, B, C and D to each question are given. The choice which you think is correct, fill that circle in front of that question with Marker or Pen ink in the answer-book. Cutting or filling two or more circles will result in zero mark in that question.

1-1-  $x = at^2$ ,  $y = 2at$  are the parametric equations of:

(a) Ellipse

(b) Circle

(c) Parabola ✓

(d) Hyperbola

2-  $\lim_{n \rightarrow \infty} \left( \frac{5n+1}{5n} \right)^n = :$

(a)  $e^{1/5}$  ✓(b)  $e^5$ (c)  $e^{-5}$ (d)  $e^{-1/5}$ 

3- If  $f(x) = \cosh x$  then  $f(x)^2 - f'(x)^2 = :$

(a) 0

(b) 1

(c)  $\frac{1}{2}$  ✓(d)  $2^2$ 

4-  $\frac{d}{dx} ((\ln x)^m)^k :$

(a)  $\frac{mk}{x} (\ln x)^{mk-1}$  ✓(b)  $\frac{k}{x^m} (\ln x)^{k-1}$ (c)  $\frac{1}{x^{mk}}$ (d)  $\frac{mk}{x}$ 

5-  $\frac{d}{dx} \left( \sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 :$

(a)  $1 - \frac{1}{2x}$ (b)  $1 + \frac{1}{x^2}$ 

(c) 0

(d)  $1 - \frac{1}{x^2}$  ✓

6- If  $y = \cos x$ ,  $u = \sin x$  then  $\frac{dy}{dx} =$  :

- (a)  $\cos x$  (b)  $-\cot x \checkmark$   
(c)  $-\tan x$  (d)  $-\operatorname{cosec} x$

7-  $a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$  is:

- (a) Maclaurin's series (b) Taylor series  
(c) Power series  $\checkmark$  (d) Binomial series

8-  $\int \tan x \, dx =$  :

- (a)  $\ln \cot x + c$  (b)  $\ln \cos x + c$   
(c)  $\ln \sin x + c$  (d)  $\ln \sec x + c \checkmark$

9-  $\int e^x \left[ \frac{1}{1+x^2} + \tan^{-1} x \right] dx =$  :

- (a)  $e^x \tan x + c$  (b)  $\frac{e^x}{1+x^2} + c$   
(c)  $e^x \sin x + c$  (d)  $e^x \tan^{-1} x + c \checkmark$

10-  $\int_0^{\pi/2} \sin^3 x \cos x \, dx =$  :

- (a)  $\frac{1}{2}$  (b)  $\frac{2}{3}$   
(c)  $\frac{1}{4} \checkmark$  (d)  $\frac{1}{9}$

11- The solution of  $\frac{dy}{dx} = -y$  is:

- (a)  $y = e^{2x}$  (b)  $y = ce^{-x} \checkmark$   
(c)  $y = e^x$  (d)  $ce^x$

12- Equation of horizontal line through (a, b) is:

- (a)  $y = a$  (b)  $y = b \checkmark$   
(c)  $x = a$  (d)  $x = b$

13- The two lines  $a_1x + b_1y = c_1$  ;  $a_2x + b_2y = c_2$  are parallel if:

- (a)  $a_1 - a_2 = 0$  (b)  $a_1 - b_1 = 0$   
(c)  $a_1b_1 - a_2b_2 = 0$  (d)  $a_1b_2 - a_2b_1 = 0 \checkmark$

- 14- The inclination of  $x = y$  is:  
(a)  $30^\circ$  (b)  $60^\circ$   
(c)  $45^\circ$  ✓ (d)  $180^\circ$
- 15- If the line  $(3x - y + 5) + k(2x - 3y - 4) = 0$  is parallel to y-axis, then  $k =$  :  
(a)  $-\frac{1}{3}$  ✓ (b)  $-\frac{1}{4}$   
(c)  $-\frac{1}{5}$  (d) 0
- 16-  $ax + b < c$  is:  
(a) Linear inequality ✓ (b) Identity  
(c) Equation (d) Not inequality
- 17- If the ends of the diameter of the circle are  $(0, 1)$  and  $(2, 3)$ , then its area is:  
(a)  $\pi$  (b)  $2\pi$  ✓  
(c)  $4\pi$  (d)  $8\pi$
- 18- The directrix of the parabola  $x^2 = -8y$  is:  
(a)  $x + 2 = 0$  (b)  $x - 2 = 0$   
(c)  $y + 2 = 0$  (d)  $y - 2 = 0$  ✓
- 19- If vectors  $2\hat{i} + \hat{j} + \hat{k}$  and  $\hat{i} - 4\hat{j} + \alpha\hat{k}$  are perpendicular, then  $\alpha =$  :  
(a) 1 (b) 2 ✓  
(c) 3 (d) 4
- 20-  $[\underline{a} \ \underline{b} \ \underline{a}]$  is equal to :  
(a) 1 (b)  $\underline{a}$   
(c) 0 ✓ (d)  $\underline{b}$

## Inter (Part-II) 2017

Mathematics

Group-II

PAPER: II

Time: 2.30 Hours

(SUBJECTIVE TYPE)

Marks: 80

## SECTION-I

2. Write short answers to any EIGHT (8) questions: 16

- (i) If  $f(x) = \sqrt{x+1}$  and  $g(x) = \frac{1}{x^2}$  find  $g$  of  $(x)$ .

Ans

$$\begin{aligned} g \text{ of } (x) &= g[f(x)] \\ &= g(\sqrt{x+1}) \\ &= \frac{1}{(\sqrt{x+1})^2} \\ g \text{ of } (x) &= \frac{1}{x+1} \end{aligned}$$

- (ii) If  $f(x) = (-x+9)^3$  find  $f^{-1}(x)$ .

Ans

Given  $f(x) = (-x+9)^3$

Let  $y = f(x)$

So,  $y = (-x+9)^3$

By solving equation for  $x$

$$y^{1/3} = -x+9$$

$$x = 9 - y^{1/3}$$

As  $y = f(x)$

So,  $f^{-1}(y) = x$

or  $f^{-1}(y) = 9 - y^{1/3}$

Replacing  $y$  by  $x$ , we get

$$f^{-1}(x) = 9 - x^{1/3}$$

- (iii) Evaluate  $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta}$ .

Ans

$$\begin{aligned} \frac{1 - \cos \theta}{\theta} &= \frac{1 - \cos \theta}{\theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} \\ &= \frac{1 - \cos^2 \theta}{\theta(1 + \cos \theta)} \\ &= \frac{\sin^2 \theta}{\theta(1 + \cos \theta)} \end{aligned}$$

$$= \sin \theta \left( \frac{\sin \theta}{\theta} \right) \left( \frac{1}{1 + \cos \theta} \right)$$

$$\begin{aligned} \therefore \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} &= \lim_{\theta \rightarrow 0} \sin \theta \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \lim_{\theta \rightarrow 0} \left( \frac{1}{1 + \cos \theta} \right) \\ &= 0(1) \left( \frac{1}{1 + 1} \right) \\ &= 0 \end{aligned}$$

(iv) If  $y = (x^2 + 5)(x^3 + 7)$  find  $\frac{dy}{dx}$ .

**Ans** Given that  $y = (x^2 + 5)(x^3 + 7)$   
 $y = x^5 + 5x^3 + 7x^2 + 35$

Differentiating with respect to  $x$ , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [x^5 + 5x^3 + 7x^2 + 35] \\ &= \frac{d}{dx} (x^5) + 5 \frac{d}{dx} (x^3) + 7 \frac{d}{dx} (x^2) + \frac{d}{dx} (35) \\ &= 5x^4 + 5(3x^2) + 7(2x) + 0 \\ \frac{dy}{dx} &= 5x^4 + 15x^2 + 14x \end{aligned}$$

(v) Find  $\frac{dy}{dx}$  if  $y^2 + x^2 - 4x = 5$ .

**Ans** Given that  $y^2 + x^2 - 4x = 5$  (1)

Differentiating both sides of (1), w.r.t. 'x'.

$$\frac{d}{dx} [y^2 + x^2 - 4x] = \frac{d}{dx} (5)$$

$$2y \frac{dy}{dx} + 2x - 4 = 0$$

$$2y \frac{dy}{dx} = 4 - 2x$$

$$\frac{dy}{dx} = \frac{4 - 2x}{2y}$$

$$\frac{dy}{dx} = \frac{2(2 - x)}{2y}$$

$$\frac{dy}{dx} = \frac{2 - x}{y}$$

(vi) Differentiate  $\sqrt{x} + \sqrt{x}$  w.r.t.  $x$ .

**Ans** Given  $y = \sqrt{x} + \sqrt{x}$

(1)

Let  $u = x + \sqrt{x}$   
Then the equation (1) becomes

$$y = \sqrt{u}$$

Differentiating w.r.t 'x'

$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx}$$

(2)

Here  $u = x + \sqrt{x}$   
Differentiate w.r.t 'x'

$$\begin{aligned}\frac{du}{dx} &= 1 + \frac{1}{2\sqrt{x}} \\ &= \frac{2\sqrt{x} + 1}{2\sqrt{x}}\end{aligned}$$

Put values in equation (2),

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2\sqrt{x + \sqrt{x}}} \cdot \frac{2\sqrt{x} + 1}{2\sqrt{x}} \\ &= \frac{2\sqrt{x} + 1}{4\sqrt{x} \sqrt{x + \sqrt{x}}}\end{aligned}$$

(vii) Differentiate  $\ln(x^2 + 2x)$  w.r.t. x.

**Ans** Let  $y = \ln(x^2 + 2x)$ , then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\ln(x^2 + 2x)] \\ &= \frac{1}{(x^2 + 2x)} \frac{d}{dx} (x^2 + 2x) \\ &= \frac{1}{x^2 + 2x} (2x + 2) \\ &= \frac{2(x + 1)}{x^2 + 2x}\end{aligned}$$

Thus,

$$\frac{d}{dx} [\ln(x^2 + 2x)] = \frac{2(x + 1)}{x^2 + 2x}$$

(viii) If  $y = \sin^{-1}(ax + b)$  find  $\frac{dy}{dx}$ .

**Ans** Let,

$$u = ax + b$$

$$\frac{du}{dx} = a$$

Then,

$$y = \sin^{-1} u$$

$$\frac{dy}{du} = \frac{1}{\sqrt{1+u^2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$$

Thus

$$\frac{d}{dx} [\sin^{-1} (ax+b)] = \frac{1}{\sqrt{1+(ax+b)^2}} \cdot a$$

(ix) If  $y = x \cos y$  find  $\frac{dy}{dx}$ .

**Ans** Given  $y = x \cos y$   
Differentiating w.r.t  $x$

$$\frac{dy}{dx} = x \frac{d}{dx} (\cos y) + \cos y \cdot \frac{d}{dx} (x)$$

$$= x (-\sin y) \cdot \frac{dy}{dx} + \cos y$$

(1)

$$\frac{dy}{dx} = -x \sin y \frac{dy}{dx} + \cos y$$

$$\frac{dy}{dx} + x \sin y \cdot \frac{dy}{dx} = \cos y$$

$$\frac{dy}{dx} (1 + x \sin y) = \cos y$$

$$\frac{dy}{dx} = \frac{\cos y}{1 + x \sin y}$$

(x) Prove that  $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ .

**Ans** Let  $y = \sin^{-1} x$

Then  $x = \sin y$

(i)

Differentiating both sides of (i) w.r.t 'x', we get

$$1 = \frac{d}{dx} (\sin y) \frac{dy}{dx}$$

$$1 = \cos y \frac{dy}{dx}$$

$$\frac{1}{\cos y} = \frac{dy}{dx}$$

$\therefore \cos y$  is positive for  $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

$$\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}} \text{ for } -1 < x < 1$$

**Define point of inflexion.**

(xi) **Ans** A point of a curve at which a change in the direction of curvature occurs, is called point of inflexion.

**Define critical point.**

(xii) **Ans** The point  $(c, f(c))$  on the graph of  $f$  is named as a critical point.

3. Write short answers to any EIGHT (8) questions: 16

(i) Find  $dy$  in  $y = x^2 + 2x$  when  $x$  changes from 2 to 1.8.

**Ans**  $y = x^2 + 2x$   
When  $x$  changes from 2 to 1.8

$$\therefore x = 2$$

$$\Delta x = 1.8 - 2 = -0.2$$

$$\begin{aligned} y &= x^2 + 2x \\ &= (2)^2 + 2(2) = 4 + 4 \\ &= 8 \end{aligned}$$

$$y = x^2 + 2x$$

$$y + \Delta y = (x + \Delta x)^2 + 2(x + \Delta x)$$

$$\Delta y = (x + \Delta x)^2 + 2(x + \Delta x) - y$$

$$\Delta y = (2 - 0.2)^2 + 2(2 - 0.2) - 8$$

$$\Delta y = 3.24 + 3.6 - 8$$

$$(\because \Delta y \rightarrow dy)$$

$$\boxed{dy = -1.16}$$

(ii) Evaluate  $\int \tan^2 x \, dx$ .

**Ans**  $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$

$$= \int \sec^2 x \, dx - \int 1 \, dx$$

$$= \int \sec^2 x \, dx - \int dx$$

$$= \tan x - x + c$$

(iii) Find  $\int a^{x^2} x \, dx$ .

**Ans** Put

$$x^2 = t, \text{ then}$$

$$\Rightarrow 2x = \frac{dt}{dx}$$

$$x \, dx = \frac{1}{2} dt$$

$$\text{Thus, } \int a^{x^2} x \, dx = \int a^t \times \frac{1}{2} dt$$

$$= \frac{1}{2} \int a^t dt$$

$$= \frac{1}{2} \frac{a^t}{\ln a} + c$$

$$= \frac{a^{x^2}}{2 \ln a} + c$$

(iv) Evaluate  $\int \frac{1}{x \ln x} dx$ .

**Ans** Let

$$\ln x = t$$

$$\text{then } \frac{1}{x} = \frac{dt}{dx}$$

$$\therefore \frac{1}{x} dx = dt$$

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{\ln x} \cdot \frac{1}{x} dx$$

$$= \int \frac{1}{t} dt$$

$$= \ln |t| + c$$

$$= \ln |\ln x| + c$$

(v) Evaluate  $\int \frac{e^x(1+x)}{(2+x)^2} dx$ .

**Ans** Let,

$$I = \int \frac{e^x(1+x)}{(2+x)^2} dx$$

$$= \int e^x \left[ \frac{1+x}{(2+x)^2} \right] dx$$

$$= \int e^x \left[ \frac{(2+x)-1}{(2+x)^2} \right] dx$$

$$= \int e^x \left[ \frac{1}{2+x} - \frac{1}{(2+x)^2} \right] dx$$

$$= \int e^x \cdot \frac{1}{2+x} dx - \int e^x \cdot \frac{1}{(2+x)^2} dx$$

$$= \int \frac{1}{2+x} e^x dx - \int e^x \cdot \frac{1}{(2+x)^2} dx$$

Integrating by parts,

$$= \frac{1}{2+x} \cdot e^x - \int e^x \frac{-1}{(2+x)^2} dx - \int e^x \cdot \frac{1}{(2+x)^2} dx$$

$$= e^x \frac{1}{2+x} + \int e^x \frac{1}{(2+x)^2} dx - \int e^x \cdot \frac{1}{(2+x)^2} dx$$

$$I = e^x \cdot \frac{1}{2+x} + c$$

(vi) Evaluate  $\int x \ln x dx$ .

**Ans** Let  $I = \int x \cdot \ln x dx$   
 $= \int \ln x \cdot x dx$

Integrating by parts,

$$= \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \left( \frac{x^2}{2} \right)$$

$$I = \frac{x^2}{2} \cdot \ln x - \frac{1}{4} x^2 + c$$

(vii) Write two properties of definite integration.

**Ans**  $\int_a^b f(x) dx$  has a definite value  $\phi(b) - \phi(a)$ , so it is called the definite integral of  $f$  from  $a$  to  $b$ .  $\phi(b) - \phi(a)$  is denoted as  $[\phi(x)]_a^b$  or  $\phi(x) \Big|_a^b$ .

(viii) Evaluate  $\int_{\pi/6}^{\pi/3} \cos t dt$ .

**Ans** Let  $I = \int_{\pi/6}^{\pi/3} \cos t dt$

$$= [\sin t]_{\pi/6}^{\pi/3}$$

$$= \sin \frac{\pi}{3} - \sin \frac{\pi}{6}$$

$$A = \int_0^4 y \, dx$$

$$A = \int_0^4 (4x - x^2) \, dx$$

$$= \left[ 4 \frac{x^2}{2} - \frac{x^3}{3} \right]_0^4$$

$$= \left[ 2x^2 - \frac{x^3}{3} \right]_0^4$$

$$= \left[ 2(4)^2 - \frac{(4)^3}{3} \right] - \left[ 2(0)^2 - \frac{(0)^3}{3} \right]$$

$$= \left[ 2(16) - \frac{64}{3} \right] - [0 - 0]$$

$$= 32 - \frac{64}{3}$$

$$= \frac{96 - 64}{3}$$

$$A = \frac{32}{3} \text{ square units}$$

(xi) **Define optimal solution.**

**Ans** The feasible solution which maximizes or minimizes the objective function is called the optimal solution.

(xii) **Define decision variables.**

**Ans** The variables used in the system of linear inequalities relating to the problems of everyday life are non-negative constraints, which play an important role for taking decision. So these variables are called decision variables.

**4. Write short answers to any NINE (9) questions: (18)**

(i) **Show that the points A(-1, 2), B(7, 5) and C(2, -6) are vertices of a right triangle.**

**Ans** Let a, b and c denote the lengths of the sides BC, CA and AB, respectively.

By the distance formula, we have

$$c = AB = \sqrt{(7 - (-1))^2 + (5 - 2)^2} = \sqrt{73}$$

$$a = BC = \sqrt{(2 - 7)^2 + (-6 - 5)^2} = \sqrt{146}$$

$$b = CA = \sqrt{(2 - (-1))^2 + (-6 - 2)^2} = \sqrt{73}$$

Clearly :  $a^2 = b^2 + c^2 \Rightarrow \sqrt{146} = \sqrt{73} + \frac{73}{\sqrt{73}}$

Thus, ABC is a right triangle with right angle at A.

(ii) Find h, such that A(-1, h), B(3, 2) and C(7, 3) are collinear.

**Ans** Given points are A(-1, h), B(3, 2) and C(7, 3).  
As we know, three points  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  are collinear if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

As given points are collinear, so

$$\begin{vmatrix} -1 & h & 1 \\ 3 & 2 & 1 \\ 7 & 3 & 1 \end{vmatrix} = 0$$

Expanding from  $R_1$

$$(-1)(2 - 3) - h(3 - 7) + 1(9 - 14) = 0$$

$$(-1)(-1) - h(-4) + 1(-5) = 0$$

$$1 + 4h - 5 = 0$$

$$4h - 4 = 0$$

$$4(h - 1) = 0$$

$$h - 1 = 0$$

$$\boxed{h = 1}$$

(iii) Convert the equation  $4x + 7y - 2 = 0$  into two intercept form.

**Ans** Given equation;

$$4x + 7y - 2 = 0$$

$$4x + 7y = 2$$

Dividing both sides by 2

$$2x + \frac{7}{2}y = 1$$

$$\frac{\frac{x}{\frac{1}{2}}}{\frac{1}{2}} + \frac{\frac{y}{\frac{2}{7}}}{\frac{2}{7}} = 1$$

(iv) Find an equation of the perpendicular bisector of the segment joining the points A(3, 5) and B(9, 8).

**Ans** Let PQ be the perpendicular bisector of the line segment AB

Then PQ passes through the mid-point Q of AB.

Co-ordinates of Q are:  $\left[ \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right]$ .

$$Q\left(\frac{3+9}{2}, \frac{5+8}{2}\right) = Q\left(\frac{12}{2}, \frac{13}{2}\right) \\ = Q\left(6, \frac{13}{2}\right)$$

So  $(x_1, y_1) = \left(6, \frac{13}{2}\right)$

Now slope of  $\overline{AB}$  is  $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 5}{9 - 3}$  \\  $= \frac{3}{6}$  \\  $= \frac{1}{2}$

Since  $\overline{PQ} \perp \overline{AB}$

So, (slope of  $\overline{AB}$ ) (slope of  $\overline{PQ}$ ) = -1

$$\left(\frac{1}{2}\right) (\text{slope of } \overline{PQ}) = -1$$

$$\text{slope of } \overline{PQ} = -2$$

or

$$m = -2$$

Hence equation of  $\overline{PQ}$  is

$$y - y_1 = m(x - x_1)$$

$$y - \frac{13}{2} = -2(x - 6)$$

$$2y - 13 = -4(x - 6)$$

$$2y - 13 = -4x + 24$$

$$4x + 2y - 13 - 24 = 0$$

$$4x + 2y - 37 = 0$$

- (v) Check whether the point  $(-4, 7)$  is above or below of the line  $6x - 7y + 70 = 0$ .

**Ans** Given points are  $(-4, 7)$

Given line is  $6x - 7y + 70 = 0$

Multiplying both sides by -1

$$-6x + 7y - 70 = 0$$

Put  $(-4, 7)$  in L.H.S of equation

$$\begin{aligned}
 & -6(-4) - 7(7) - 70 \\
 & = 24 - 49 - 70 \\
 & = 95 > 0
 \end{aligned}$$

So the points  $(-4, 7)$  lies above the given line.

- (vi) Find an equation of tangent to the circle  $x^2 + y^2 = 2$  parallel to the line  $x - 2y + 1 = 0$ .

**Ans**

$$\begin{aligned}
 x^2 + y^2 &= 2 \\
 \text{we know } x^2 + y^2 &= r^2 \\
 r^2 &= 2
 \end{aligned}$$

$$r = \pm \sqrt{2}$$

Let equations of required tangents be:

$$y = mx \pm r \sqrt{1 + m^2} \quad (i)$$

Given line

$$\begin{aligned}
 x - 2y + 1 &= 0 \\
 -2y &= -x - 1 \\
 2y &= x + 1 \\
 y &= \frac{1}{2}x + \frac{1}{2}
 \end{aligned}$$

$\therefore$  Slope of required tangents

$$m = \frac{1}{2}$$

By putting the values in (i),

$$\begin{aligned}
 y &= \frac{1}{2}x \pm \sqrt{2} \sqrt{1 + \frac{1}{4}} \\
 &= \frac{x}{2} \pm \sqrt{2} \sqrt{\frac{5}{4}} \\
 &= \frac{x \pm \sqrt{10}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \therefore 2y &= x \pm \sqrt{10} \\
 \Rightarrow x - 2y \pm \sqrt{10} &= 0
 \end{aligned}$$

- (vii) Define focal chord of parabola.

**Ans** A chord passing through the focus of a parabola is called a focal chord of the parabola.

- (viii) Find centre and vertices of ellipse  $\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1$ .

**Ans** Given ellipse:  $\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1$

Centre of the above ellipse is:

$$x = 0, \quad y = 0$$

or

$$x = 1, \quad y = -1, \text{ i.e., } (1, -1) \text{ is centre.}$$

Vertices of the above ellipse is  $(0, \pm a) = (0, \pm 3)$ .

$$x = 0, \quad y = \pm 3$$

i.e.,

$$x = 1, \quad y = -1 \pm 3$$

or

$$(1, -4) \text{ and } (1, 2) \text{ are the vertices.}$$

- (ix) Find vertices and equation of directories of hyperbola  $x^2 - y^2 = 9$ .

**Ans** Given equation is:

$$x^2 - y^2 = 9$$

$$\frac{x^2}{9} - \frac{y^2}{9} = 1$$

(i)

From equation (1), we see that  $a = 3, b = 3$

$$\text{Vertices} = (\pm a, 0) = (\pm 3, 0)$$

$$c^2 = a^2 + b^2 = 9 + 9$$

$$c = 3\sqrt{2}$$

The equation of directories are:

$$x = \pm \frac{a}{e} = \pm \frac{a^2}{ae} = \pm \frac{a^2}{c}$$

$$x = \pm \frac{9}{3\sqrt{2}} = \pm \frac{3}{\sqrt{2}}$$

Now  $e = \frac{c}{a} = \frac{3\sqrt{2}}{3} = \sqrt{2}$

- (x) Find a vector of length 5, in the direction of opposite that of  $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$ .

**Ans** Given vector is:

$$\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$$

Then,

$$\begin{aligned} |\underline{v}| &= \sqrt{(1)^2 + (-2)^2 + (3)^2} \\ &= \sqrt{1 + 4 + 9} \end{aligned}$$

$$|\underline{v}| = \sqrt{14}$$

If  $\underline{u}$  is a unit vector in the direction of  $\underline{v}$ , then

$$\underline{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$\underline{u} = \frac{\underline{i} - 2\underline{j} + 3\underline{k}}{\sqrt{14}}$$

$$\underline{u} = \frac{1}{\sqrt{14}} (\underline{i} - 2\underline{j} + 3\underline{k})$$

Now a vector of length 5 in the direction opposite to that of  $\vec{v}$  is

$$\begin{aligned} -5 \underline{u} &= -5 \left[ \frac{1}{\sqrt{14}} (\underline{i} - 2\underline{j} + 3\underline{k}) \right] \\ &= \frac{-5}{\sqrt{14}} (\underline{i} - 2\underline{j} + 3\underline{k}) \\ &= \frac{-5}{\sqrt{14}} \underline{i} + \frac{10}{\sqrt{14}} \underline{j} - \frac{15}{\sqrt{14}} \underline{k} \end{aligned}$$

- (xi) Find a scalar  $\alpha$ , so that the vectors  $2\underline{i} + \alpha\underline{j} + 5\underline{k}$  and  $3\underline{i} + \underline{j} + \alpha\underline{k}$ , are perpendicular.

**Ans** Let  $\underline{u} = 2\underline{i} + \alpha\underline{j} + 5\underline{k}$

and  $\underline{v} = 3\underline{i} + \underline{j} + \alpha\underline{k}$

It is given that  $\underline{u}$  and  $\underline{v}$  are perpendicular

$$\therefore \underline{u} \cdot \underline{v} = 0$$

$$\Rightarrow (2\underline{i} + \alpha\underline{j} + 5\underline{k}) \cdot (3\underline{i} + \underline{j} + \alpha\underline{k}) = 0$$

$$6 + \alpha + 5\alpha = 0$$

$$6\alpha = -6$$

$$\therefore \alpha = -1$$

- (xii) If  $\underline{a} + \underline{b} + \underline{c} = 0$ , then prove that  $\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$ .

**Ans** Given

$$\underline{a} + \underline{b} + \underline{c} = 0 \quad (1)$$

Taking cross product on both sides with  $\underline{a}$ .

$$\underline{a} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{a} \times 0$$

$$\underline{a} \times \underline{a} + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = 0$$

$$\underline{a} \times \underline{b} + \underline{a} \times \underline{c} = 0$$

$$\therefore \underline{a} \times \underline{a} = 0$$

$$\underline{a} \times \underline{b} = -\underline{a} \times \underline{c}$$

$$\underline{a} \times \underline{b} = \underline{c} \times \underline{a} \quad (2)$$

Taking cross product on both sides of (1) with  $\underline{b}$ ,

$$\underline{b} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{b} \times 0$$

$$\underline{b} \times \underline{a} + \underline{b} \times \underline{b} + \underline{b} \times \underline{c} = 0$$

$$\underline{b} \times \underline{a} + \underline{b} \times \underline{c} = 0$$

$$\therefore \underline{b} \times \underline{b} = 0$$

$$\underline{b} \times \underline{a} = -\underline{b} \times \underline{c}$$

$$-\underline{a} \times \underline{b} = -\underline{b} \times \underline{c}$$

$$\underline{a} \times \underline{b} = \underline{b} \times \underline{c}$$

(3)

From (2) and (3),

$$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$$

- (xiii) Prove that the vectors  $\underline{i} - 2\underline{j} + 3\underline{k}$ ,  $-2\underline{i} + 3\underline{j} - 4\underline{k}$  and  $\underline{i} - 3\underline{j} + 5\underline{k}$ , are coplanar.

**Ans**

$$\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k}$$

$$\underline{v} = -2\underline{i} + 3\underline{j} - 4\underline{k}$$

$$\underline{w} = \underline{i} - 3\underline{j} + 5\underline{k}$$

If  $\underline{u}$ ,  $\underline{v}$  and  $\underline{w}$  are coplanar then  $\underline{u} \cdot \underline{v} \times \underline{w} = 0$ 

Now,

$$\underline{u} \cdot \underline{v} \times \underline{w} = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}$$

Expanding from  $R_1$ ,

$$= 1(15 - 12) + 2(-10 + 4) + 3(6 - 3)$$

$$= 3 + 2(-6) + 3(3)$$

$$= 3 - 12 + 9$$

$$\underline{u} \cdot \underline{v} \times \underline{w} = 0$$

## SECTION-II

**NOTE: Attempt any THREE (3) questions.**

- Q.5.(a) Prove that  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$ . (5)

**Ans**

By the binomial theorem, we have

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \dots$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \dots$$

when  $n \rightarrow \infty$ ,  $\frac{1}{n}$ ,  $\frac{2}{n}$ ,  $\frac{3}{n}$ , ... all tend to zero

$$\therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

$$= 1 + 1 + 0.5 + 0.166667 + \dots$$

$$= 2.718281 \dots$$

As approximate value of  $e$  is  $= 2.718281$

$$\therefore \lim_{x \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

(b) If  $y = \tan (p \tan^{-1} x)$ , show that  $(1 + x^2)y_1 - p(1 + y^2) = 0$ . (5)

**Ans** Given

$$y = \tan (p \tan^{-1} x)$$

Differentiate w.r.t  $x$

$$\frac{dy}{dx} = \sec^2 (p \tan^{-1} x) \cdot \frac{d}{dx} (p \tan^{-1} x)$$

$$y_1 = \sec^2 (p \tan^{-1} x) \cdot p \cdot \frac{1}{1 + x^2}$$

Multiplying both sides by  $(1 + x^2)$ ,

$$(1 + x^2) y_1 = p \sec^2 (p \tan^{-1} x)$$

$$= p [1 + \tan^2 (p \tan^{-1} x)]$$

$$(1 + x^2) y_1 = p(1 + y^2)$$

$$(1 + x^2) y_1 - p(1 + y^2) = 0$$

Q.6.(a) Show that  $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln (x + \sqrt{x^2 - a^2}) + c$ . (5)

**Ans** For Answer see Paper 2015 (Group-I), Q.6.(a).

(b) Find the point three-fifth of the way along the line segment from  $A(-5, 8)$  to  $B(5, 3)$ . (5)

**Ans** For Answer see Paper 2016 (Group-I), Q.4.(ii).

Q.7.(a) Evaluate  $\int_1^3 \frac{x^2 - 2}{x + 1} dx$ . (5)

**Ans** Let  $I = \int_1^3 \frac{x^2 - 2}{x + 1} dx$

$$= \int_1^3 \left[ x - 1 - \frac{1}{x + 1} \right] dx$$

$$\begin{aligned}
 &= \int_1^3 (x-1) dx - \int_1^3 \frac{1}{x+1} dx \\
 &= \left[ \frac{(x-1)^2}{2} \right]_1^3 - [\ln(x+1)]_1^3 \\
 &= \frac{1}{2} [(x-1)^2]_1^3 - [\ln(x+1)]_1^3 \\
 &= \frac{1}{2} [(3-1)^2 - (1-1)^2] - [\ln(3+1) - \ln(1+1)] \\
 &= \frac{1}{2} [2^2 - 0] - [\ln 4 - \ln 2] \\
 &= \frac{1}{2} (4) - \ln \left( \frac{4}{2} \right) \\
 I &= 2 - \ln 2
 \end{aligned}$$

(b) Graph the feasible region subject to the following constraints: (5)

$$2x - 3y \leq 6$$

$$2x + y \geq 2, \quad x \geq 0, \quad y \geq 0$$

**Ans** The graph of  $2x - 3y \leq 6$  is the closed half-plane on the origin side of  $2x - 3y = 6$ . The portion of the graph of system  $2x - 3y \leq 6$ ,

$$x \geq 0, y \geq 0$$

is shown as shaded region in figure (a).

When  $x = 0$

$$2(0) - 3y = 6$$

$$-3y = 6$$

$$y = -2$$

$$P_1(0, -2)$$

When  $y = 0$

$$2x - 3(0) = 6$$

$$2x = 6$$

$$x = 3$$

$$P_2(3, 0)$$

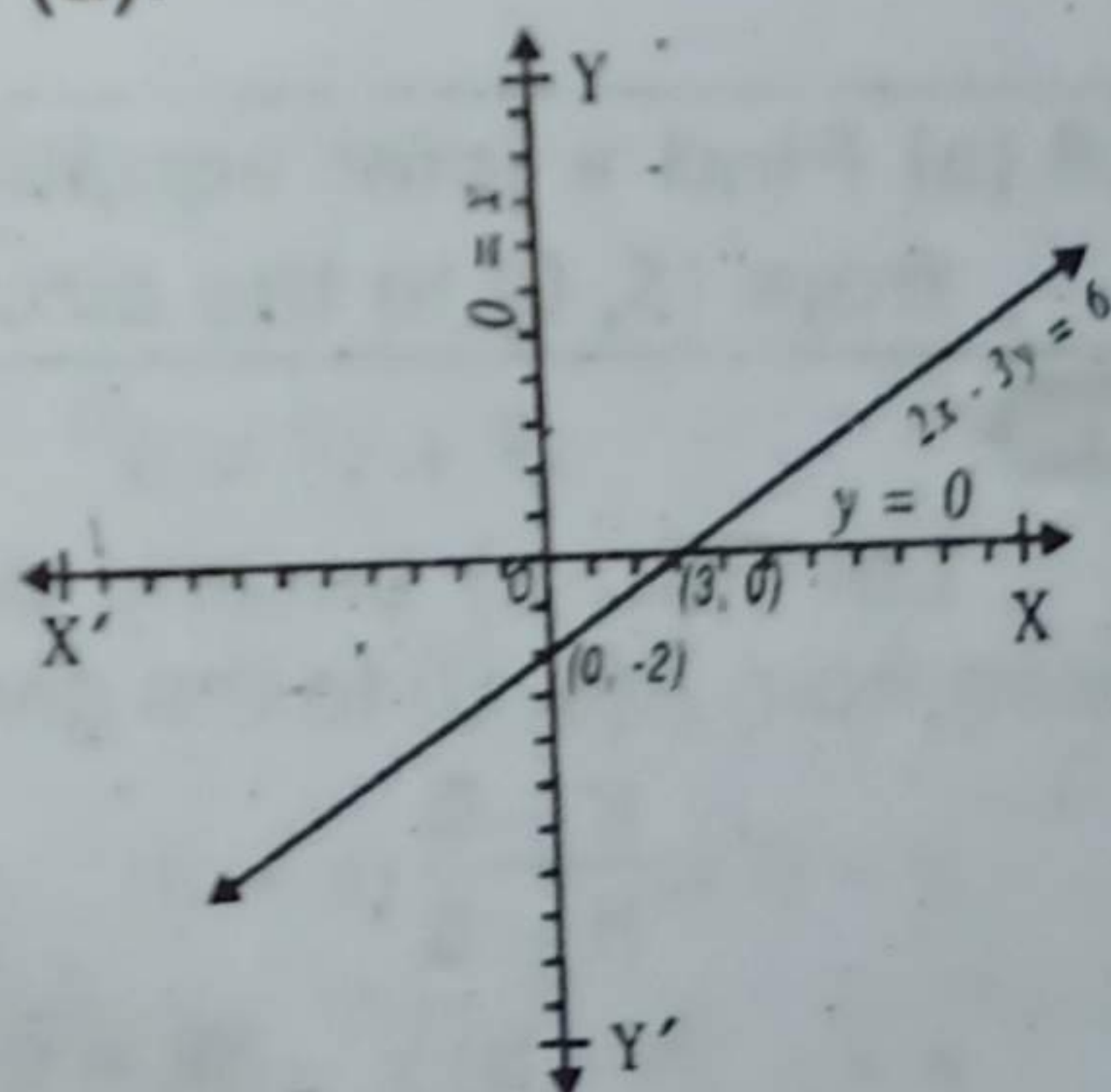


Fig. (a)

The graph of  $2x + y \geq 2$  is the closed half-plane not on the origin side of  $2x + y = 2$ . The portion of the graph of the system  $2x + y \geq 2$ ,

$$x \geq 0, y \geq 0$$

is displayed as shaded region in figure (b).

When  $x = 0$

$$2(0) + y = 2$$

$$y = 2$$

$$P_1(0, 2)$$

When  $y = 0$

$$2x + 0 = 2$$

$$2x = 2$$

$$x = 1$$

$$P_2(1, 0)$$

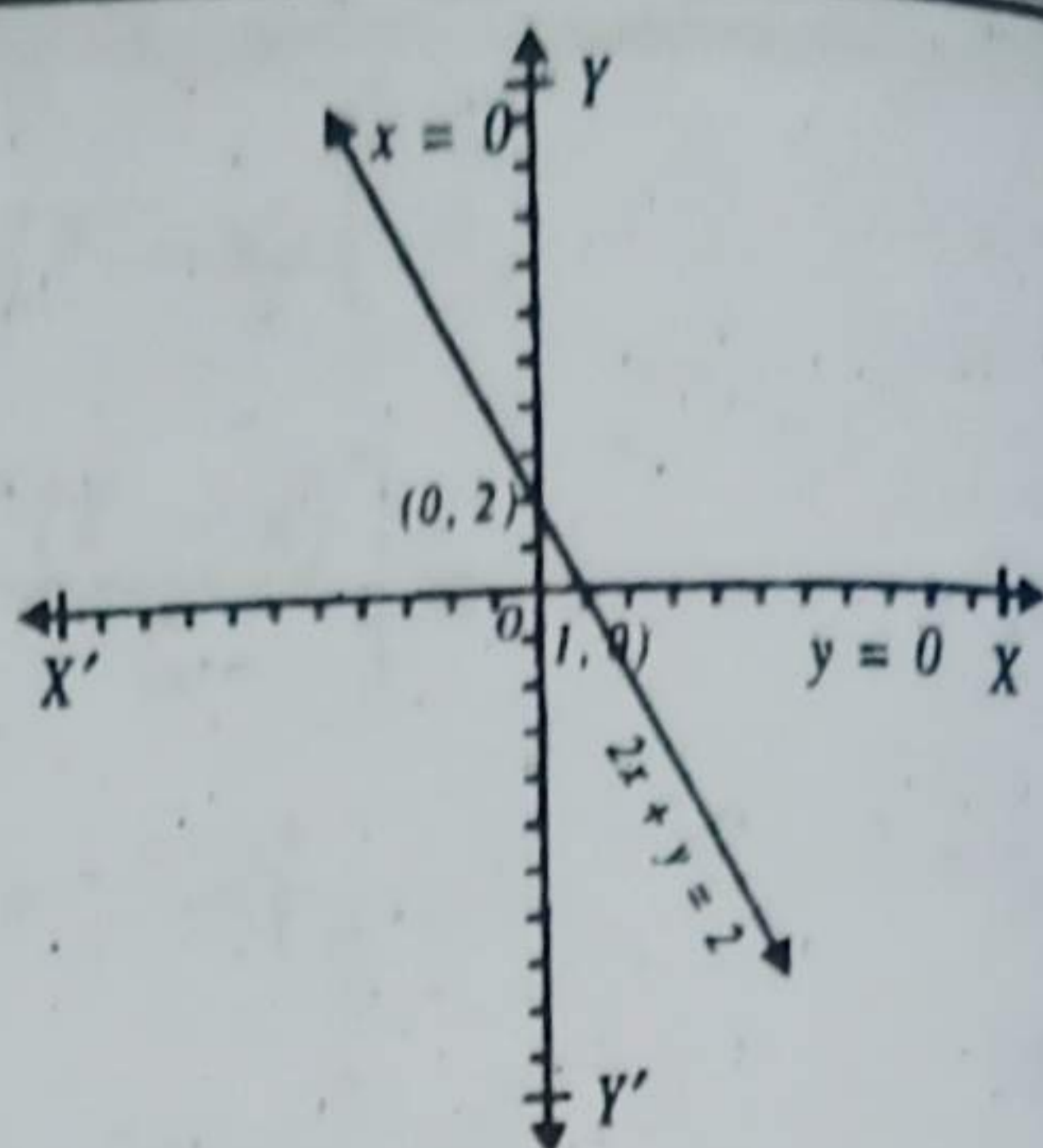


Fig. (b)

The graph of the system

$$2x - 3y \leq 6, 2x + y \leq 2,$$

$$x \geq 0, y \geq 0$$

is the intersection of the graphs shown in figures (a) and (b) and it is partially displayed in figure (c) as shaded region.

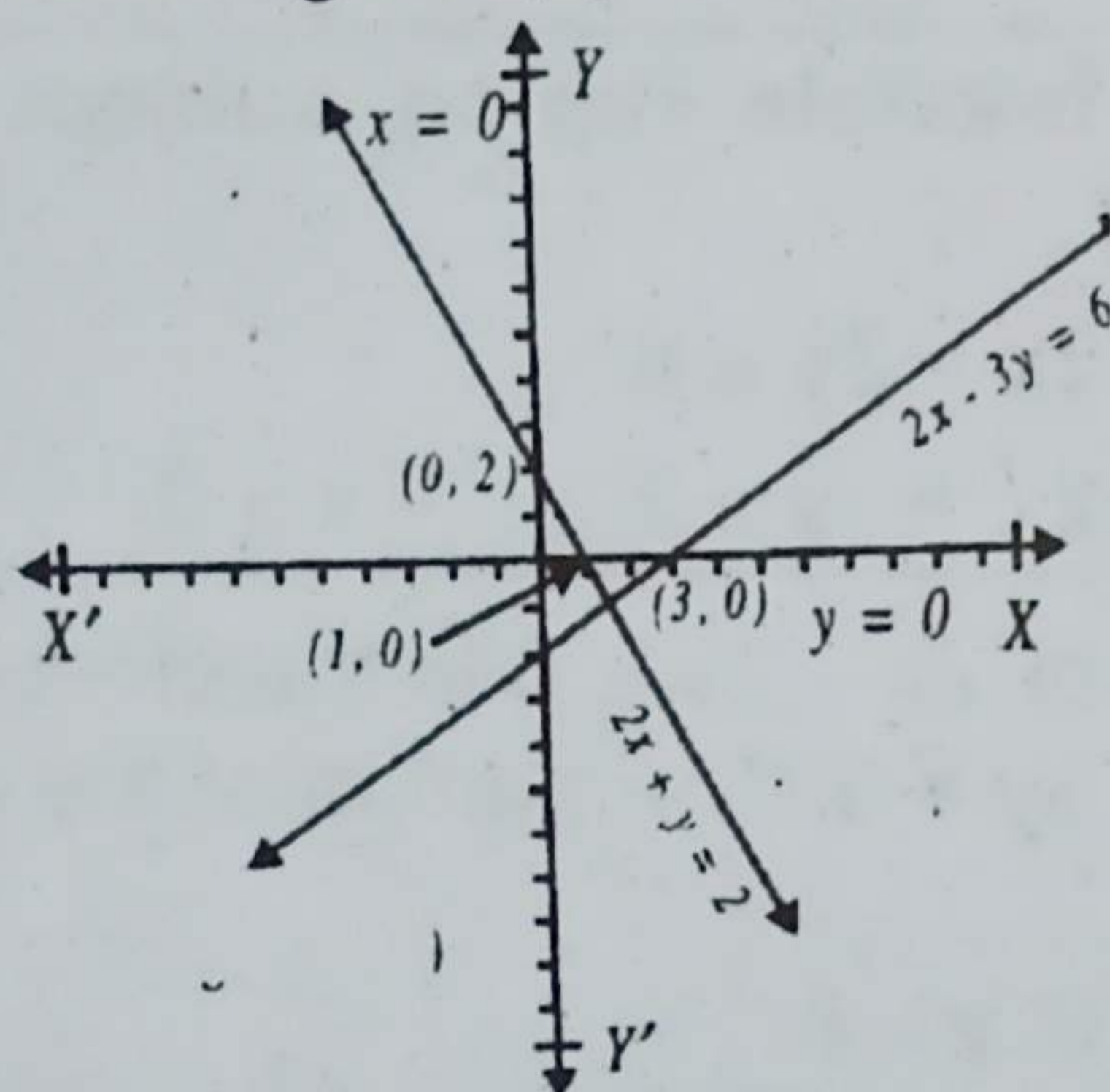


Fig. (c)

**Q.8.(a)** Find a joint equation to the pair of tangents drawn from  $(5, 0)$  to the circle  $x^2 + y^2 = 9$ . (5)

**Ans**

$$x^2 + y^2 = 9$$

(1)

Let  $P(h, k)$  be any point on either of the two tangents drawn from  $A(5, 0)$  to the given circle (1). Equation of  $PA$  is

$$y - 0 = \frac{k - 0}{h - 5} (x - 5)$$

$$kx - (h - 5)y - 5k = 0$$

(2)

Since (2) is tangent to the circle (1), the perpendicular distance of (2) from the centre of the circle equals the radius of the circle.

$$\text{i.e., } \frac{|-5k|}{\sqrt{k^2 + (h - 5)^2}} = 3$$

or  $25k^2 = 9[k^2 + (h - 5)^2]$   
 $16k^2 - 9(h - 5)^2 = 0$

Thus  $(h, k)$  lies on

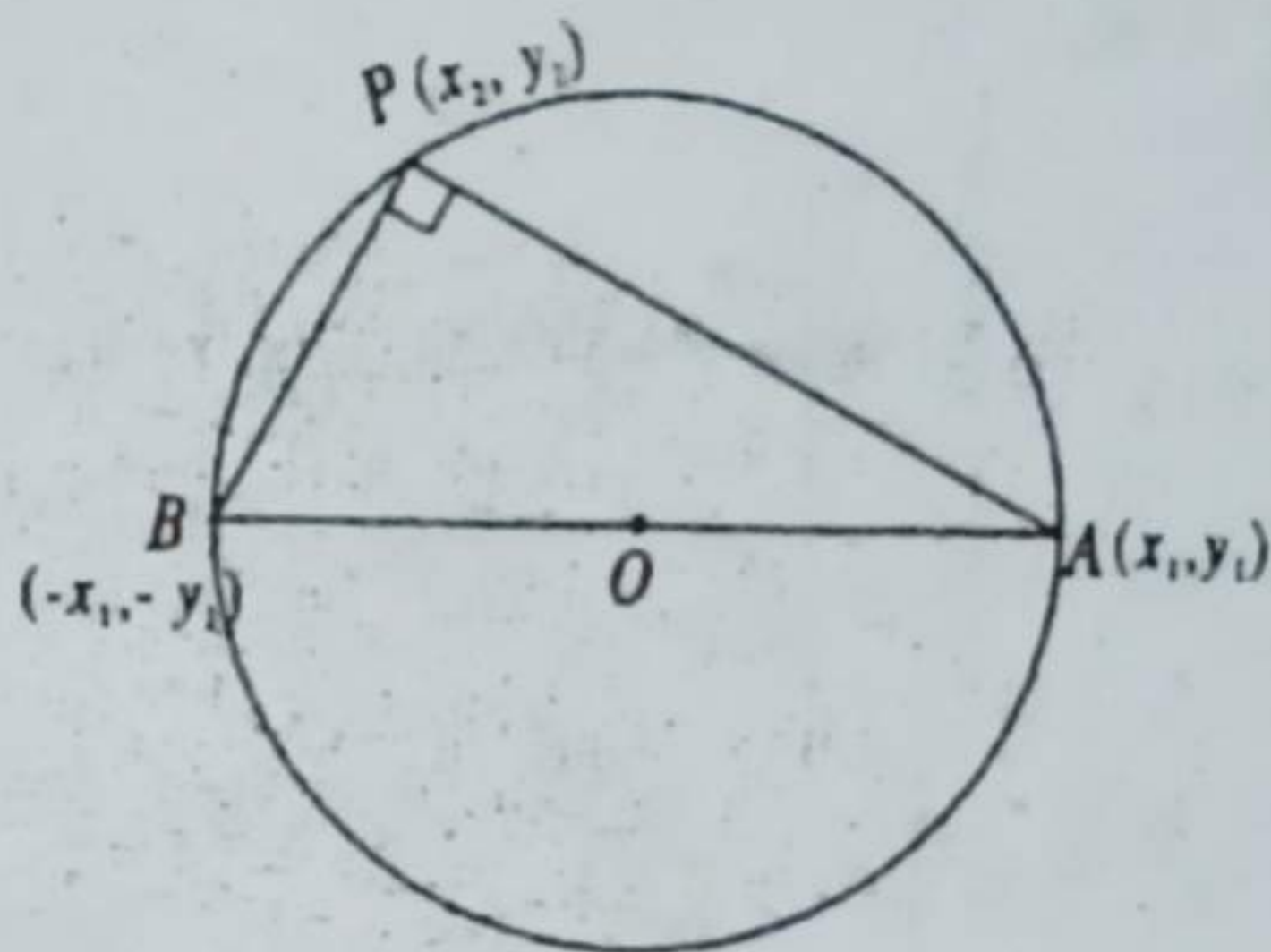
$$9(x - 5)^2 - 16y^2 = 0 \quad (3)$$

But  $(h, k)$  is any point of either of the two tangents.

Hence (3) is the joint equations of the two tangents.

(b) **Prove that the angle in a semi-circle is a right angle.**

**Ans** Let  $x^2 + y^2 = a^2$  be a circle, with centre  $O$ . Let  $AOB$  be any diameter of the circle and  $P(x_2, y_2)$  be any point on the circle.



We have to show that  $m\angle APB = 90^\circ$ .

Suppose the coordinates of  $A$  are  $(x_1, y_1)$ .

Then  $B$  has coordinates

$$(-x_1, -y_1)$$

$$\text{Slope of } AP = \frac{y_1 - y_2}{x_1 - x_2} = m_1, \text{ say}$$

$$\text{Slope of } BP = \frac{y_1 + y_2}{x_1 + x_2} = m_2, \text{ say}$$

$$m_1 m_2 = \frac{y_1^2 - y_2^2}{x_1^2 - x_2^2} \quad (1)$$

Since  $A(x_1, y_1)$  and  $P(x_2, y_2)$  lie on the circle, we have

$$\left. \begin{aligned} x_1^2 + y_1^2 &= a^2 \Rightarrow x_1^2 = a^2 - y_1^2 \\ x_2^2 + y_2^2 &= a^2 \Rightarrow x_2^2 = a^2 - y_2^2 \end{aligned} \right] \quad (2)$$

Substituting the values of  $x_1^2$  and  $x_2^2$  from (2) into (1), we

get

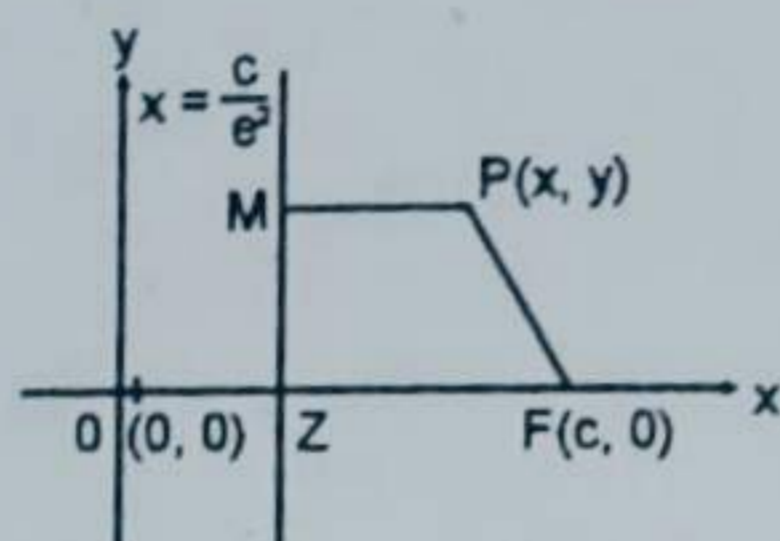
$$m_1 m_2 = \frac{y_1^2 - y_2^2}{(a^2 - y_1^2) - (a^2 - y_2^2)} = \frac{y_1^2 - y_2^2}{-(y_1^2 - y_2^2)} = -1$$

Thus  $AP \perp BP$  and so  $m\angle APB = 90^\circ$ .

**Q.9.(a) Derive equation of hyperbola in standard form. (5)**

**Ans** Let  $F(c, 0)$  be the focus with  $c > 0$  and  $x = \frac{c}{e^2}$  be the directrix of the hyperbola. Also let  $P(x, y)$  be a point on the hyperbola, then by definition

$$\frac{|PF|}{|PM|} = e$$



$$\begin{aligned} \text{i.e., } (x - c)^2 + y^2 &= e^2 \left( x - \frac{c}{e^2} \right)^2 \\ &= e^2 \left( x^2 + \frac{c^2}{e^4} - 2x \frac{c}{e^2} \right) \\ &= e^2 x^2 + \frac{c^2 e^2}{e^4} - \frac{2x c e^2}{e^2} \end{aligned}$$

$$e^2 x^2 - 2cx - x^2 + 2cx - y^2 = c^2 - \frac{c^2}{e^2}$$

$$\text{or } x^2 - 2cx + c^2 + y^2 = e^2 x^2 - 2cx + \frac{c^2}{e^2}$$

$$\text{or } x^2(e^2 - 1) - y^2 = c^2 \left( 1 - \frac{1}{e^2} \right) = \frac{c^2}{e^2} (e^2 - 1) \quad (2)$$

Let us set  $a = \frac{c}{e}$ , so that (2) becomes

$$x^2(e^2 - 1) - y^2 - a^2(e^2 - 1) = 0$$

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

$$\text{or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (3)$$

where  $b^2 = a^2(e^2 - 1) = c^2 - a^2 \quad \therefore c = ae$   
(3) is standard equation of the hyperbola.